A Connection Between Vector Bundles over Smooth Projective Curves and Representations of Quivers

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Introduction

- Quiver
- Representation of Quiver
- Vector Bundle on Curve
• Directed graph
• Multi-edges and self-loops allowed

Example

1 → 2 → 3

1

4

2

3

4
Representation of Quiver

- Vector Space for each vertex
- Linear transformation for each edge

Example

\[ \begin{align*}
1(\mathbb{R}) & \xrightarrow{\alpha: \mathbb{R} \to \mathbb{R}^2} 2(\mathbb{R}^2) \\
2(\mathbb{R}^2) & \xrightarrow{\beta: \mathbb{R}^2 \to \mathbb{R}} 3(\mathbb{R}) \\
3(\mathbb{R}) & \xrightarrow{\gamma: \mathbb{R}^2 \to \mathbb{R}^2} 2(\mathbb{R}^2) \\
4(\mathbb{R}^3) & \xrightarrow{\delta: \mathbb{R}^3 \to \mathbb{R}^2} 2(\mathbb{R}^2) \\
4(\mathbb{R}^3) & \xrightarrow{\epsilon: \mathbb{R}^3 \to \mathbb{R}} 3(\mathbb{R}) \\
4(\mathbb{R}^3) & \xrightarrow{\zeta: \mathbb{R}^3 \to \mathbb{R}} 3(\mathbb{R})
\end{align*} \]
Vector Bundle on Curve

- Curve $C$
- Vector space $\mathbb{F}^k$
- Copy of $\mathbb{F}^k$ for each $c \in C$
- Continuously varying
Example (Open Cylinder)

One copy of $\mathbb{R}$ for each point on circle
Example (Open Mobius Strip)

One copy of $\mathbb{R}$ for each point on circle.
Conjecture (Schiffmann, 2016)

For any $g \in \mathbb{N}^+$ and $r \in \mathbb{N}$ and $d \in \mathbb{Z}$,

$$A_{g,r,d}(0) = A_{\Sigma g,r}(1).$$
Definition

A partition $\lambda$ is a finite non-increasing sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0.$$  

The size of $\lambda$ is $|\lambda| = \lambda_1 + \cdots + \lambda_l$.

The length of $\lambda$ is $l(\lambda) = l$.

A flat partition is one in which all elements are equal.

Example

$$\lambda = (4, 4, 2, 1) \quad \lambda = (3, 3, 3, 3)$$
Definition

For any partition $\pi$ and any $g$, we define the rational function

$$B_\pi = \sum_{\pi_0, \ldots, \pi_s} q^{\pi_0 \cdot l(\pi_0)} (-1)^s \prod_{i=0}^{s} \frac{q^{(g-1)\langle \pi_i, \pi_i \rangle}}{b_{\pi_i}(q^{-1})}$$

where $\pi_0$ may be the empty partition but $\pi_1, \ldots, \pi_s$ are all nonempty, and $\pi_0 \cup \pi_1 \cup \cdots \cup \pi_s = \pi$.

Conjecture

For all partitions $\pi$, $B_\pi$ is a polynomial.
**Theorem**

For all flat partitions $\pi$, $B_\pi$ is a polynomial.
Proving Polynomialness

Example

\[ \frac{1}{1-q} = 1 + q + q^2 + q^3 + \ldots \]

Example

\[ \frac{1}{1-q^2} = 1 + q^2 + q^4 + q^6 + \ldots \]

Example

\[ \frac{1 - q^2}{1-q} = (1 - q^2)(1 + q + q^2 + q^3 + \ldots) \]

\[ = (1 + q + q^2 + q^3 + \ldots) - (q^2 + q^3 + q^4 + q^5 + \ldots) \]

\[ = 1 + q \]
Proving Polynomialness

\[ \pi = (a, a, a, \ldots, a) \]
\[ l(\pi) = b \]
\[ (-1)^b q^{(g-1)a} B_\pi = c_0 q^0 + c_1 q^1 + c_2 q^2 + \ldots \]

**Theorem**

For some \( N \), for all \( n \geq N \), the coefficient \( c_n = 0 \).

**Corollary**

\[ \text{deg}(-1)^b q^{(g-1)a} B_\pi = b(b + 1)(g - 1)a + b(a - 1). \]
Combinatorial Interpretation

\[ c_n = \sum_p \text{sign}(p), \text{ where } p = (p^0, p^1, \ldots, p^s) \text{ and } \text{sign}(p) = (-1)^s \]

Example \((a = 2, b = 5, g = 2, n = 50)\)

\[ p = ((15, 10, 5), (11, 6)) \]

\begin{itemize}
  \item \(p^1, \ldots, p^s\) are nonempty
  \item sum of lengths of partitions is \(b = 5\)
  \item sum of sizes of partitions, plus \((a - 1)l(p^0), \text{ is } n = 50\)
  \item each \(p^i\) is a \(d\)-stair partition for \(d = 2(g - 1)a + 1 = 5\).
\end{itemize}
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is **d-stair** if every consecutive difference is at least $d$, and $\lambda_l \geq d$.

**Example**

$p^0 = (15, 10, 5)$ is 5-stair.

$p^1 = (11, 6)$ is 5-stair.
Need to cancel out all tuples $\lambda$ of the same size

\[ n = (a - 1)l(\lambda^0) + |\lambda^0| + |\lambda^1| + \cdots + |\lambda^s| \]

for large enough $n$.

Idea: pair each tuple $\lambda$ with another tuple of opposite sign - i.e. with one more or one less partition.

1. Pair \{tuples with zeroth partition nonempty\} with part of \{tuples with zeroth partition empty\}
2. Pair off remaining elements of \{tuples with zeroth partition empty\}
Zeroth Partition Nonempty

Example \((a = 4, b = 3, g = 2, n = 40)\)

\[
((9), (19, 9)) \rightarrow ((\), (12), (19, 9))
\]

Example \((a = 4, b = 3, g = 2, n = 42)\)

\[
((18, 9), (9)) \rightarrow ((\), (21, 12), (9))
\]

We pair

\[
\{ p : |p^0| > 0 \}
\]

with

\[
\{ p : |p^0| = 0 \wedge p^1_{l(p^1)} \geq d + a - 1 \} \text{ where } d = 2(g - 1)a + 1
\]
What’s left is

\[ \{ p : |p^0| = 0 \land p^1_{l(p^1)} < d + a - 1 \} \]

Two operations:
- Unroll
- Tuck
We apply whichever comes first.
Example \((a = 2, d = 5)\)

- every partition is a singleton
- differences less than \(d = 2(g - 1)a + 1\)
- first value less than \(d + a - 1\)

\[\therefore \text{size is bounded by}\]

\[
|p^0| + |p^1| + \cdots + |p^b|
\leq 0 + (d + a - 2) + \cdots + (d + a - 2 + (b - 1)(d - 1))
= b(b + 1)(g - 1)a + b(a - 1).
\]
Either

- size is small
- everything cancels out

**Theorem**

*For all flat partitions* $\pi$, $B_\pi$ *is a polynomial.*
Future Directions

Conjecture (General Case of $B_{\pi}$)

For all partitions $\pi$, $B_{\pi}$ is a polynomial.

\[
\begin{align*}
p_0^1 & \quad p_1^1 & \cdots & \quad p_s^1 \\
p_0^2 & \quad p_1^2 & \cdots & \quad p_s^2 \\
p_0^3 & \quad p_1^3 & \cdots & \quad p_s^3 \\
\vdots & \quad \vdots & \ddots & \quad \vdots
\end{align*}
\]
Acknowledgements

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For any questions about quivers and vector bundles and curves, the chances are quite high that I don’t know the answer.

**Example**

Q: What is the analogy between representations of quivers and vector bundles on curves?
A: I don’t know.