Posets and their Incidence Algebras

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Outline

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What is a poset?

Definition
A poset \((P, \preceq)\) is a set \(P\) which has a partial order \(\preceq\) imposed on it, and \(\preceq\) is reflexive, antisymmetric, and transitive. That is,

- For \(t \in P\), \(t \preceq t\).
- \(s \preceq t\) and \(t \preceq s\) means \(t = s\).
- \(s \preceq t\) and \(t \preceq u\) means \(s \preceq u\).

Note. This does not mean that for all \(s, t \in P\) we have \(s \preceq t\) or \(t \preceq s\). There need not be a comparison under \(\preceq\), and when this is the case we call \(s, t\) incomparable. This is shown by writing \(s \parallel t\).
Drawing a poset

Definition
The **Hasse diagram** of a poset is a graph where $s \in P$ is a vertex, and $s, t$ are connected by an edge if $s \preceq t$ and there is no $u$ such that $s \prec u \prec t$. If $s \preceq t$, $s$ is placed above $t$ in the Hasse diagram.

We can say that posets $P$ and $Q$ are **isomorphic** if we can create an order preserving bijection $\phi$ between them. This is denoted $P \cong Q$.

\[ \text{Figure 1: Isomorphic Hasse diagrams of a 4-element poset.} \]
Definition
A closed interval \([a, b]\) is the set of all \(p \in P\) such that \(a \leq p \leq b\). The set of closed intervals is denoted \(\text{Int}(P)\).

Definition
A chain is a collection of elements \(t_i\) of a poset \(P\) such that \(t_0 \prec t_1 \ldots \prec t_n\).

Definition
A multichain is a collection of elements \(t_i\) of a poset \(P\) such that \(t_0 \leq t_1 \ldots \leq t_n\).
Examples

What do chains and intervals look like in a Hasse diagram?
The Incidence Algebra

Definition

The **incidence algebra** on a poset $P$ over a field $K$, denoted $I(P, K)$ is a $K$-algebra over the vector space of functions

$$f : \text{Int}(P) \to K$$

equipped with the bilinear product called convolution, denoted $\ast$, given by

$$f \ast g([s, u]) := \sum_{s \leq t \leq u} f([s, t])g([t, u]).$$

**Note.** It is usually not necessary to use fields where $\text{ch}(K) \neq 0$, so using $K = \mathbb{C}$ is often sufficient when working with $I(P, K)$. From this point on, we will use $I(P, \mathbb{C})$. 

The identity in $I(P, \mathbb{C})$

The delta function $\delta$ is given by

$$\delta([s, t]) = \begin{cases} 1, & s = t \\ 0, & s \neq t \end{cases}.$$

It is a two sided identity under convolution: $f * \delta = \delta * f = f$. 
The zeta function $\zeta$ is given by

$$\zeta([s,t]) = 1.$$ 

We define the Möbius function $\mu$ is defined by

$$\mu \ast \zeta = \delta.$$ 

**Proposition**

The function $\mu$ can be computed by setting $\mu([s,s]) = 1$, and

$$\mu([s,u]) = - \sum_{s \leq t < u} \mu([s,t]).$$
Why does the zeta function matter?

One major reason is that the zeta function is extremely useful for counting chains. Denote $f^n$ as $f \ast f \ldots \ast f$ $n$ times for $f \in I(P, \mathbb{C})$. Then we have the following propositions:

**Proposition**

$$\zeta^n([s, t]) = \sum_{s \leq s_1 \ldots \leq s_{n-1} \leq t} 1.$$ 

**Proposition**

The function $(\zeta - \delta)^n([s, t])$ counts the number of chains in the interval $[s, t]$. 
Möbius Inversion Theorem

Theorem (Möbius Inversion)

Let $P$ be a poset where for every $t \in P$ the order ideal $\Lambda_t$ is finite. For $f, g : P \to \mathbb{C}$ we have

$$g(t) = \sum_{s \preceq t} f(s) \zeta([s, t]) \iff f(t) = \sum_{s \preceq t} g(s) \mu([s, t])$$

for all $t \in P$.

Note. The order ideal $\Lambda_t$ is defined as the set of elements less than $t$. 
Take $\mathbb{C}^P$, the set of functions $f : P \to \mathbb{C}$. Then $I(P, \mathbb{C})$ acts on $\mathbb{C}^P$ via

$$(f\mathcal{I})(t) = \sum_{s \leq t} f(s)\mathcal{I}([s, t])$$

for $f \in \mathbb{C}^P, \mathcal{I} \in I(P, \mathbb{C})$. Then we have the equivalent statement to Möbius inversion

$$g = f\zeta \iff f = g\mu.$$ 

**Note.** Here, $I(P, \mathbb{C})$ acts on the right. We get a ‘dual’ theorem by acting on the left.
Example

Let $B_n$ be the poset with underlying set $\{S \mid S \subseteq [n]\}$ and partial order $\preceq$ given by $S \preceq T$ if $S \subseteq T$. We then have

$$\mu_{B_n} = (-1)^{|T-S|}.$$  

Applying Möbius inversion, we obtain

$$g(T) = \sum_{S \subseteq T} f(S) \iff f(T) = \sum_{S \subseteq T} g(S)(-1)^{|T-S|}.$$  

Setting $f(T) = f_=(T)$ where $f_=$ counts objects having exactly the properties in $T$, and $g(T) = g_\leq(T)$ where $g_\leq$ counts objects having at most the properties in $T$ we obtain the principle of Inclusion-Exclusion.
Example

Take the ‘divisor poset’ $D_n$, which has an underlying set of
\{d : d \in \mathbb{N}, d|n\} and partial order $\leq$ given by $a \leq b$ if $a|b$. We
obtain that

$$\mu_{D_n}([s, t]) = \begin{cases} (-1)^k & \text{if } t/s = \prod_i p_i \text{ for distinct primes } p_i \\ 0 & \text{otherwise} \end{cases}$$

from which it follows

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} g(d) \mu_{D_n}([d, n]).$$
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