Defining the Jones Polynomial in terms of the Tutte Polynomial

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Knot Definitions
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What is a knot?
Knot Definitions

A knot is a loop of string which satisfies the following restrictions:

It is a closed and smooth curve in 3 space

It does not intersect itself anywhere
Knot Definitions

A Knot embedding is a smooth injection $f : S^1 \rightarrow \mathbb{R}^3$.

Knot Embedding Equivalency:

We define 2 knot embeddings to be equivalent if they are isotopic.
Knot Definitions

- A knot isotopy is a smooth map $h: S^1 \times I \to \mathbb{R}^3$ that maps each circle to a knot embedding.

- The top knot embedding is isotopic to the bottom knot embedding.

- Knot isotopy is an equivalence relation.

- A knot is an equivalence class of knot embeddings.
Knot Definitions

Knot Diagram:

Project the knot onto a flat plane

We can use \( \times \) for crossings, to show over and under.

A knot diagram is alternating if following the strands results in an over under pattern. Every knot has an alternating representation.

However, we need to pick a plane to project it onto. There cannot be a perpendicular to the plane that contains 3 points on the knot. Also, the derivative of the knot cannot be a perpendicular to the plane at any point.
Knot Definitions

Taking 2 different knot embeddings or projection directions can result in different knot diagrams for the same knot.

We need an equivalence relation for knot diagrams expressing when 2 knot diagrams represent the same knot.

These are the 3 Reidemeister moves.
From Alternating Knots to Graphs

- Shade regions like before
  - Planar Duals
  - Use the one not containing infinity
- Place a vertex in each shaded region
- Place an edge through each crossing
Tutte Polynomial

- Recurrence: $T(G) = T(G-e) + T(G/e)$
- Base Case
- Need to check:
  - this associates a well-defined polynomial to a graph
  - gives a Knot invariant
Tutte Polynomial

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<th>Tutte Polynomial Definition</th>
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1. Let $T \in G$ be the subset of all spanning trees in $G$. Then $\chi_G(x, y) = \sum_T x^r y^s$ where $r$ is the number of externally active edges and $y$ is the number of internally active edges.

2. For all isthmuses $e_j$ we have $\chi_G(x, y) = x \cdot \chi_{G'_{j}}(x, y)$ and for all edges which are also loops $e_k$ we have $\chi_G(x, y) = y \cdot \chi_{G''_{k}}(x, y)$. For all other edges $e_j$ which are neither loops nor isthmuses we have $\chi_G(x, y) = \chi_{G'_{j}}(x, y) + \chi_{G''_{j}}(x, y)$.
Graph Theory Definitions

- Connected Graph
- Cycle
- Tree
- Subgraph
- Spanning Tree
- Random Labeling
- Isthmus
Graph Theory Definitions

- Cyc(T,e)
- Externally Active
Graph Theory Definition

- Cut(T,e)
- Internally Active
Tutte Polynomial Invariant

- Proof
  - Take 2 edges $E_1, E_2$
  - Only possible change in activity if $E_1$ is in $\text{cyc}(T, E_2)$
  - Casework
Tutte Polynomial Recurrence

● Proof
  ○ Assume edge taken is edge maximally labeled
    ■ Possible by last proof
  ○ Clearly Satisfies Relationship
Conclusion

- $J(K; t) = f(G; t)*T(G; t, t^{-1})$
  - Jones Polynomial = weight times Tutte
  - Weight determined by graph, link in terms of $t$
Famous Open Questions

● Complex Knots with Jones Polynomial equal to 1?
  ○ Links already proven

● There is a difficult prove that an \((m,n)\) torus has jones polynomial
  \[ t^{(m-1)(n-1)/2} \left( 1-t^{m+1}-t^{n+1}+t^{m+n} \right)/1-t^2. \] Is there a simple proof?
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