Signatures of Stable Multiplicity Spaces in Restrictions of Representations of Symmetric Groups

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Abstract

Representation theory is a way of studying complex mathematical structures such as groups and algebras by mapping them to linear actions on vector spaces. Recently, Deligne proposed a new way to study the representation theory of finite groups by generalizing the collection of representations of a sequence of groups indexed by positive integer rank to an arbitrary complex rank, creating an abelian tensor category. In this project, we focused on the case of the symmetric groups S_n, the groups of permutations of n objects. Elements of the Deligne category Rep S_t can be constructed by taking a stable sequence of S_n representations for increasing n and interpolating the associated formulas to an arbitrary complex number t. In this project, we studied the case of restriction multiplicity spaces $V_{\lambda,\rho}$, counting the number of copies of an irreducible representation V_ρ of S_{n-k} in the restriction $\mathrm{Res}^{S_n}_{S_{n-k}} V_\lambda$ of an irreducible representation of S_n. We found formulas for norms of orthogonal basis vectors in these spaces, and ultimately for signatures (the number of basis vectors with positive norm minus the number with negative norm), an invariant that multiplies over tensor products and has important combinatorial connections.

1 Introduction

Groups are fundamental objects in mathematics and physics. They are the mathematical abstraction of physical symmetry and their definition encodes the three main properties of symmetry: doing nothing is a symmetry, two symmetries can be composed to get a new symmetry and for every operation that is a symmetry of your space, the inverse operation is also a symmetry.

Classically, the first groups that were studied, and this was even before the idea of a group was formalized, were the symmetric groups S_n: the collection of permutations of the set $\{1, \dots, n\}$. The first reason why these groups are so important is that every finite group can be embedded inside a symmetric group for sufficiently large n ([3]). More important, however, is the applications the symmetric group has in combinatorics ([10]), in topology ([1]) and, of course, in algebra (for example, the lack of solvability of the alternating subgroup of the symmetric group S_5 is used in Galois' proof that general formulas solving a quintic polynomial in radicals do not exist [3]).

Many of these applications pass through the representation theory of the symmetric group. A standard idea in mathematics is to simplify the study of complicated objects by using linear approximations to the object. In the case of groups, this is done by studying representations, or linear actions of the group on a vector space that preserve the group multiplication. Finite groups like the symmetric group have particularly elegant representation theory. Any representation can be broken up into a sum of irreducible representations, those that do not contain smaller representations within, and the irreducible representations are classified by their character, i.e., the trace function they induce on group elements ([7]).

The irreducible representations of the symmetric group have been extensively studied (see [7, Ch 5] for a full primer). The irreducibles of S_n correspond to partitions of n, and their characters form a special basis of the space of symmetric functions in n variables called Schur functions. These two facts are foundational results in the field of algebraic combinatorics ([10]). Additionally, the representation theory of S_n has deep connections with the representation theory of the group of invertible $N \times N$ matrices GL_N ([7]) , a central object in the theory of Lie groups that plays a major role in quantum mechanics, geometry and modern algebra (see [9], [8] for the tip of the iceberg).

In this paper, we study a different aspect of the representation theory of symmetric groups that has only begun to be studied more recently. We take a sequence of S_n representations, one for each sufficiently large n, that satisfy a suitable compatibility condition (which we will call stability), and examine stable properties of this sequence. One of the first ways in which this question was formalized and studied was in [1] where the authors constructed a notion of FI-modules that encoded the stability of a sequence of S_n-representations. Here, we take a slightly different approach and formalize stability using the idea of a Deligne category $\operatorname{Rep} S_t$.

Deligne categories $\operatorname{Rep} S_t$ are a generalization of the collection of representations of S_n to an arbitrary complex number t ([5], [4]). They were originally defined by Deligne ([2]) as an example of a category that looks similar to representations of a group but cannot actually be realized as such. Nowadays, they are used to study the stable representation theory of the symmetric group. At values of t other than positive integers, this category is semisimple, i.e., like representations of a finite group, every object can be broken up into irreducibles, the smallest possible objects. Irreducibles in the category are labeled by all partitions λ (not of any fixed size), where λ corresponds to a stable sequence of irreducible S_n representations formed by fixing a partition and increasing the largest element. It turns out that a remarkable number of properties of the representations in the stable sequence, such as dimension (see [4]), stabilize to polynomials in n for sufficiently large n, which can be interpolated to give values in $\operatorname{Rep} S_t$.

In this paper, we study restriction multiplicity spaces in $\operatorname{Rep} S_t$. Fix some positive integer k and assume $t \in \mathbb{R}$ is not a positive integer. There is a restriction functor (see [4]) from $\operatorname{Rep} S_t$ to $\operatorname{Rep} S_{t-k}$. Since objects in $\operatorname{Rep} S_{t-k}$ are semisimple, an irreducible λ will break down into the direct sum of partitions ρ tensored with a multiplicity space $V_{\lambda,\rho}$, a vector space whose dimension counts the number of times ρ appears inside λ. For real values t that are not positive integers, objects in $\operatorname{Rep} S_t$ are equipped with non-degenerate bilinear pairings $\lambda \otimes \lambda \to \mathbb{R}$, unique up to taking scalar multiples. Taking ratios, we obtain non-degenerate bilinear pairings on $V_{\lambda,\rho}$ and since the latter is a real vector space, such a pairing is classified by its signature, which is the number of orthogonal basis vectors with positive self-product minus the number with negative self-product. Our ultimate goal is produce formulas for this signature.

We begin with definitions and background information in Section 2. In Section 3.1, we compute the norms in the case where the largest element is the same in λ and ρ. We show that in this case the norm is the same for all basis vectors up to a positive constant. In Section 3.2, we compute the norms in the case where some boxes are added to the first row. We prove that in this case the roots of the rational functions for the norms come in pairs that differ by 1, so the signature is positive definite for sufficiently large of small n. In Section 4.1, we give an example of how these norm formulas can be used to compute signatures in a special case, and in Section 4.2, we prove that among the squares added to the smaller Young diagram to form the larger, only one per column can impact the norm, which simplifies the computation of signature formulas in many cases.

2 Definitions

The *symmetric group* S_n is the group of permutations of n objects. A *representation* of a group is a homomorphism from the group to linear maps on a vector space. An *irreducible representation* is a representation where the vector space has no subspace invariant under the group action. The irreducible representations of S_n correspond to partitions, or equivalently, to combinatorial objects called Young diagrams.

Definition. A *Young diagram* is an arrangement of n boxes into rows such that each row contains at most as many boxes as the row above it (Fig. 1a-b). A *Young tableau* is a Young diagram with one of the numbers $1, 2, ..., n$ in each of the boxes (Fig. 1c). A permutation acts on a Young tableau by permuting its entries.

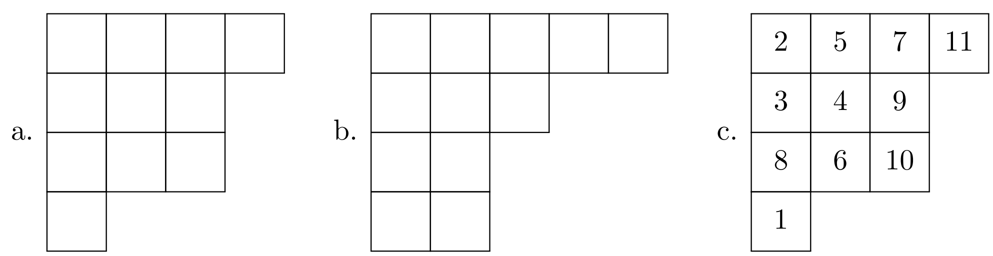


Figure 1: Young diagrams for (a) $n = 11$ and (b) $n = 12$; (c) Young tableau for $n = 11$.

Definition. A *standard Young tableau* is a Young tableau where the numbers in each row and column are increasing from left to right and top to bottom (Fig. 2a). The *standard row tableau* is the tableau in which the boxes are labelled moving across each row (Fig. 2b), and the *standard column tableau* is the tableau in which boxes are labelled moving down each column (Fig. 2c).

a.

1	3	4	7
2	5	8	
6	10	11	
9			

b.

1	2	3	4
5	6	7	
8	9	10	
11			

c.

1	5	8	11
2	6	9	
3	7	10	
4			

Figure 2: (a) Standard tableau; (b) Row tableau; (c) Column tableau.

Theorem 1. *[6] Each irreducible representation of S_n uniquely correspond to a Young diagram, with a basis given by the standard Young tableaux for that Young diagram.*

In our case, we study restrictions of irreducible representations of S_n to irreducible representations of S_{n-k}. The restrictions of an irreducible representation of S_n to S_{n-1} are formed by removing a box from the Young tableau for the larger representation. Restrictions from S_n to S_{n-k} correspond to paths determined by the order in which we remove the k boxes from the larger tableau to form the smaller one.

Each path corresponds to a standard Young tableau by filling in the numbers $1, 2, ..., n-k$ as in the standard column tableau for our representation of S_{n-k}, then filling in the numbers $n-k+1, ..., n$ in the order in which the boxes were added (Fig. 3) [12].

$$\begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & & \end{array} \quad \rightarrow \quad \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & 7 & \end{array} \quad \rightarrow \quad \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & 8 \\ 3 & 7 & \end{array}$$

Figure 3: Sample path from a representation of S_6 to S_8.

There is a unique inner product up to scaling that is invariant under the group action. We let the standard column tableau for the representation of S_n has norm 1. The basis given above is orthogonal under this inner product. Norms under this inner product are determined by the locations of boxes in the tableau. We label the rows of a Young tableau $1, 2, ...$ from top to bottom, and the columns $1, 2, ...$ from left to right.

Definition. The *content* a_m of m is the number of the column in which m is located minus the number of the row. A_m is the content of m in the standard column tableau.

Definition. The *admissible transpositions* for a standard Young tableau are the transpositions $s_i = (i,\ i+1)$, where $1 \le i \le n-1$, provided that i and $i+1$ are not in the same row or column.

Note that these transpositions will always take a standard Young tableau to another standard Young tableau. The transpositions are useful because they can be used to get from *any* standard Young tableau to any other standard Young tableau.

Definition. The *length* of a standard Young tableau is the minimum number of admissible transpositions necessary to get to that tableau from the standard row tableau.

If this product of transpositions is the permutation s, the length of s is equal to to the number of inversions, i.e. $\ell(s) = \#\{(i,j) \in \{1, 2, ..., n\} | i < j, s(i) > s(j)\}$. The transposition $(i,\ i+1)$ increases the number of inversions by one if i and $i+1$ were not initially inverted, and will decrease this number by one if they were (since (i,j) were inverted in the initial tableau iff $(i+1,j)$ were inverted in the resulting tableau) [11].

Because admissible transpositions can get between any two basis vectors, the actions of admissible transpositions on basis vectors allow us to compute the corresponding actions of all group elements. The following formula tells how an admissible transposition acts on a basis vector and changes its norm:

Theorem 2. *[11] For two standard Young tableaux $T' = s_i T$ such that $\ell(T') > \ell(T)$, the corresponding basis vectors v_T and v'_T satisfy $s_i \cdot v_T = v'_T + \dfrac{a}{a_{i+1} - a_i} v_T$.*

Corollary. *It follows that $\|v_T\|^2 = \dfrac{(a_{i+1} - a_i)^2}{(a_{i+1} - a_i)^2 - 1} \cdot \|v_{T'}\|^2$. These fractions are always positive.*

These norms are important because they can be interpolated to cases where n is an arbitrary real number, and then are sometimes negative. The number of times the norms are negative has useful combinatorial significance in the corresponding Deligne categories.

Definition. The *signature* for an inner product of a representation is the number of basis vectors with positive norm minus the number with negative norm.

This signature not depend on the choice of orthogonal basis. An inner product is *positive definite* if the norm is always positive and *negative definite* if it is always negative.

3 Norms

We fix the shapes of our larger and smaller Young tableaux, but their sizes vary by adding boxes to the first row. Let k be the fixed difference between the sizes of the larger and smaller tableaux, and n the variable size of the larger tableau. Let m be the size of the second smallest Young tableau of this larger shape, i.e. the tableau where the first row is one square longer than the second. In this section, we compute formulas for these norms, first in the case where no added boxes are in the first row, in which case the norm is positive definite, and then in the case where some added boxes are in the first row, in which case the signatures are nontrivial.

3.1 Norms in Non-First Row Case

In this section, we consider the case where the boxes added to the smaller tableau are not in the first row. Let $c_0 > c_1 > ..., > c_{k-1}$, (i.e. from right to left and bottom to top), be the k entries the added boxes contain in the standard column tableau for the larger Young diagram (Fig. 4).

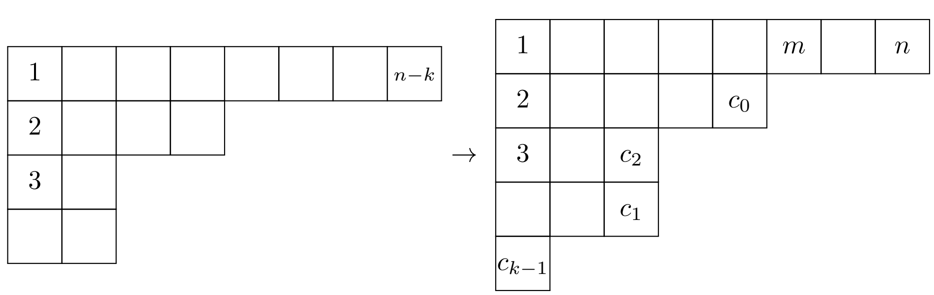


Figure 4: Standard column tableaux for Young diagrams for S_{n-k} and S_n.

To get from the standard column tableau for S_n to one of the basis vectors, we apply sequences of admissible transpositions first to move n into the desired position, then $n-1$, $n-2$, ..., $n-k+1$ (Fig. 5). If multiple boxes are added to the same column, the larger ones must be in the lower row.

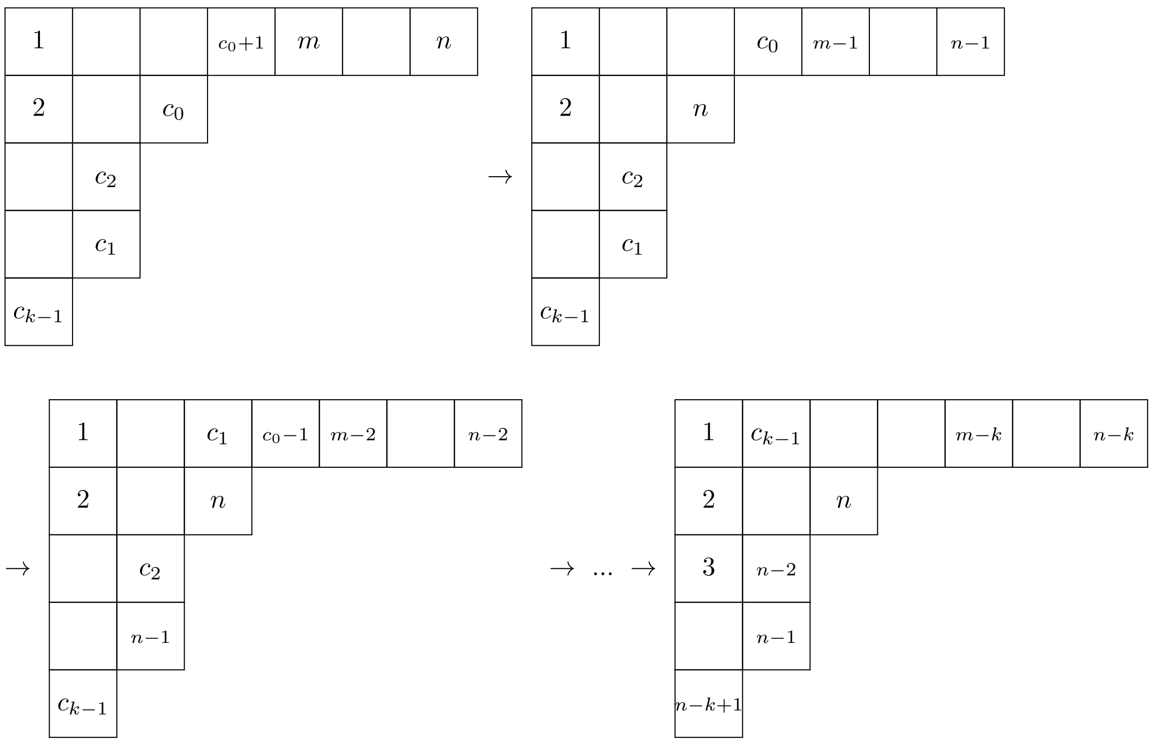
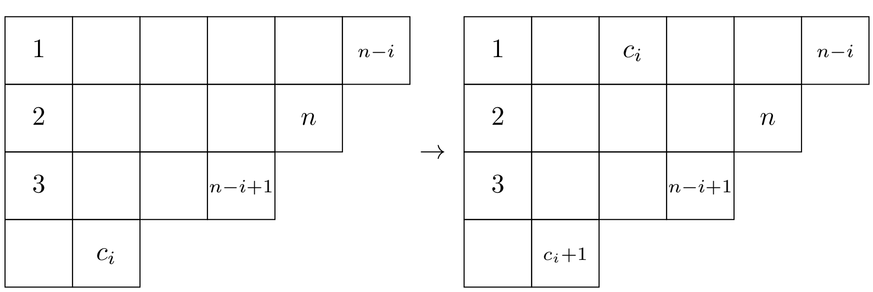


Figure 5: Admissible transpositions are applied to move $n, ..., n-k+1$ into position.

Let c_i be the entry in the standard column tableau in the box where we wish to place $n-i$. Then, assuming the numbers greater than $n-i$ are already in the desired positions, we move $n-i$ into place via the following product of transpositions (Fig. 6):

$$(n-i-1,\ n-i)(n-i-2,\ n-i-1)...(c_i,\ c_i+1).$$



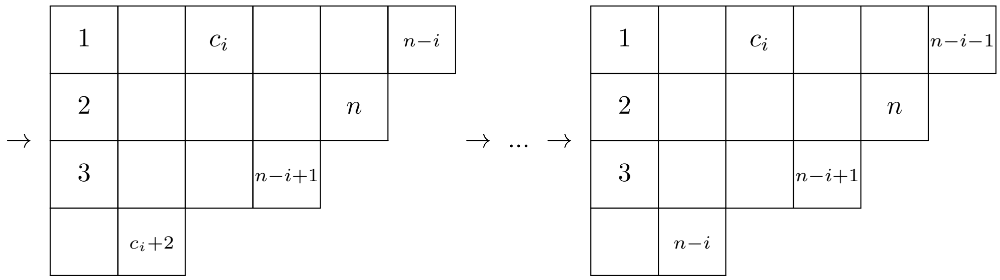


Figure 6: Admissible transpositions to move $n-i$ into place.

This product does not affect anything to the left of and above c_i, moves $n-i$ into position, and moves all the numbers from c_i to $n-i-1$ one position down or to the next column.

Now, we will see what happens to the length of our tableau after these transpositions.

Lemma 3. *All of these transpositions decrease the length of the resulting permutation.*

Proof. When the transposition $(j,\ j+1)$ is applied, j is in the position initially occupied by c_i, and $j+1$ is somewhere to the right of j in the Young tableau, i.e. earlier in the permutation corresponding to the Young tableau. Therefore, j and $j+1$ were initially inverted, and so this transposition decreases the length by one. $\square$

Next, we note that the sign of the norm is the same for all basis vectors.

Lemma 4. *Changing our choice of basis vector only multiplies the norm by a positive constant, i.e., the norm is always either positive definite or negative definite.*

Proof. All the basis vectors can be obtained from each other by a sequence of admissible transpositions of $n-k+1, ..., n-1, n$. The contents of these numbers are simply $A_{c_0}, A_{c_1}, ..., A_{c_{k-1}}$, which are independent of n, so when these transpositions are applied, the norm is multiplied by a positive constant. This implies that the norm is always positive definite or negative definite. $\square$

Now, we can find how this product of transpositions affects the norm.

Theorem 5. *The ith product of transpositions multiplies the norm by a positive constant times*

$$\frac{n-m+A_m-A_{c_i}}{n-m+A_m-A_{c_i}+1}.$$

Thus, the overall norm of the basis vectors is a positive constant times

$$\prod_{i=0}^{k-1}\frac{n-m+A_m-A_{c_i}}{n-m+A_m-A_{c_i}+1}.$$

Proof. When the transposition $(j,\ j+1)$ is applied, j is in the square initially occupied by c_i, so $a_j = A_{c_i}$ (Fig. 7) At this point, $n-i+1, n-i+2, ..., n$ have already been moved out of the first row to their final locations, so the numbers in the rightmost section of the first row have been moved over i positions and now are $m-i+1, m-i+2, ..., n-i$.

1		c_i			$n-i$
2				n	
3			$n-i+1$		
		$j+1$			
	j				

$\rightarrow$

1		c_i			$n-i$
2				n	
3			$n-i+1$		
		j			
	$j+1$				

Figure 7: Action of transposition $(j,\ j+1)$ when $n-i+1, ..., n$ are in position.

The transposition $(j,\ j+1)$ multiplies the norm by $\dfrac{(a_{j+1}-a_j)^2}{(a_{j+1}-a_j)^2-1}$. At this point, j is in the location initially occupied by c_i, so $a_j = A_{c_i}$. Now, either $a_{j+1} = A_{j+1}$ (if $j+1$ is smaller than all the larger c_i's), or else $j+1$ is larger than some of the c_i's and has already been moved several positions down or to the right. In either case, for all $j+1 < m-i+1$, $j+1$ is still in the left portion of the tableau, to the right of the extended first row, and the content of $j+1$ is dependent only on the shape of the tableau and not on n. Thus, the transpositions through $(m-i-1,\ m-i)$ multiply the norm by some constant independent of n, and this constant is positive because we are always multiplying by positive fractions.

Now, the numbers $m-i+1, m-i+2, ..., n-i$ have all been moved i positions over and so are in the first row, in the positions initially occupied by $m, m+1, ..., n$. Thus, their contents $a_{m-i+1}, a_{m-i+2}, ..., a_{n-i}$ are equal to the contents of $m+1, ..., n$ in the standard column tableau, i.e. $A_{m+1}, A_{m+2}, ..., A_n$, or, since they are all in the same row, $1+A_m, 2+A_m, ..., n-m+A_m$. Using Theorem 2 and the fact that our lengths are always decreasing, this sequence of transpositions multiplies the norm by

$$\prod_{j=m-i}^{n-i-1} \frac{(a_{j+1}-a_j)^2}{(a_{j+1}-a_j)^2-1} = \prod_{j=m}^{n-1} \frac{(j+1-m+A_m-A_{c_i})^2}{(j+1-m+A_m-A_{c_i})^2-1}$$

$$= \prod_{j=m}^{n-1} \frac{(j+1-m+A_m-A_{c_i})^2}{(j-m+A_m-A_{c_i})(j+2-m+A_m-A_{c_i})}.$$

This product telescopes, leaving $\dfrac{1+A_m-A_{c_i}}{A_m-A_{c_i}} \cdot \dfrac{n-m+A_m-A_{c_i}}{n-m+A_m-A_{c_i}+1}$. The first term is a positive constant, and the second is the desired function. Applying this formula inductively as we move each $n-i$ to the location of c_i gives the norm of the basis vector as a constant times

$$\prod_{i=0}^{k-1} \frac{n-m+A_m-A_{c_i}}{n-m+A_m-A_{c_i}+1}.$$

$\square$

3.2 Norms in First Row Case

In this section, we consider the case where some of the added boxes are in the first row. In this case, the norm is no longer positive definite in general, although it is for sufficiently large or small values of n because the roots still come in pairs that differ by 1.

Suppose j of the k boxes are added to the end of the first row, and the other $k-j$ added boxes have entries $c_1, c_2, ..., c_{k-j}$ in the standard column tableau. Let the $k-j$ numbers not in the first row be $n-d_1 > n-d_2 > ... > n-d_{k-j}$, and suppose they are moved to positions occupied $c_1, c_2, ..., c_{k-j}$, respectively, in the standard column tableau.

Theorem 6. *The norm of the resulting basis vector is* $\displaystyle\prod_{i=1}^{k-j} \frac{n-d_i+i-1-m+A_m-A_{c_i}}{n-d_i+i-m+A_m-A_{c_i}}$.

Proof. We proceed inductively, as in Theorem 4, first moving $n-d_1$ into the desired position, then $n-d_2$, and so on (Fig. 8). Note that $c_1, c_2, ..., c_{k-j}$ are not necessarily decreasing and could be in any order. Each time one of the $n-d_i$'s is moved into place, the numbers to the right of c_i are moved one position down or to the next column.

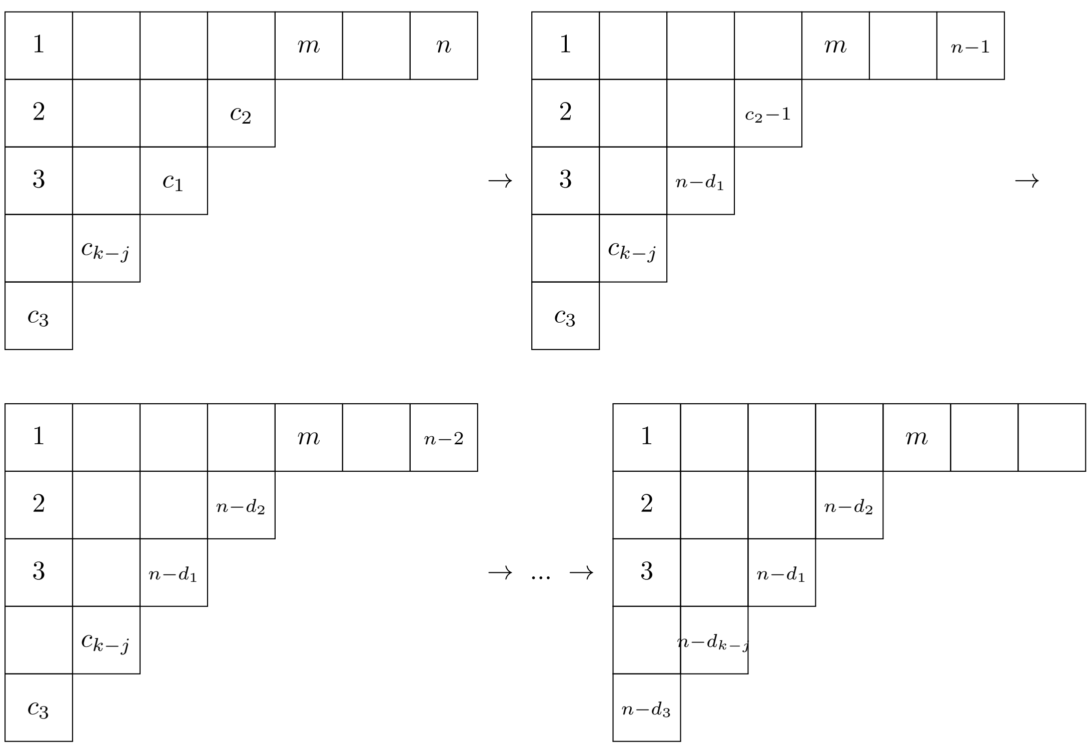


Figure 8: Each $n-d_i$ is moved into position.

To move $n-d_i$ into position, we apply the product of transpositions $(n-d_i-1,\ n-d_i)(n-d_i-2,\ n-d_i-1)...(c_i,\ c_i+1)$, unless c_i has already been moved to the right, in which case we have a slightly different starting value, but this does not affect the norms (Fig. 9). The lengths are decreasing, as before, so the ith sequence of transpositions multiplies the norm by

$$\prod_{l=c_i}^{n_i-1} \frac{(a_{l+1}-a_l)^2}{(a_{l+1}-a_l-1)(a_{l+1}-a_l+1)}.$$

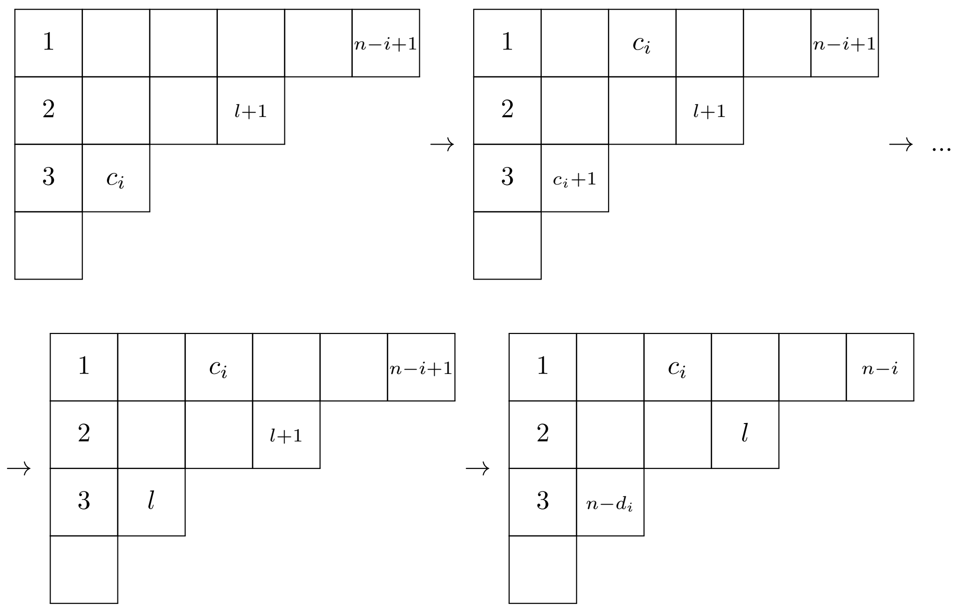


Figure 9: Sequence of transpositions to move $n - d_i$ into position.

This product is a constant from $l = c_i$ to $l = m - 1$, since the contents of these numbers depend only on the fixed shape of the tableau. For the remaining $n_i - m$ terms, when the transposition $(l,\ l+1)$ is applied, l is in the square to which we will be moving $n - d_i$ and so has content A_{c_i}. The content of l in the standard tableau was $A_l = l - m + A_m$, since l is in the first row. However, when each one of $n-d_1, n-d_2, ..., n-d_{i-1}$ was moved into position, all the other numbers greater than or equal to m and less than them were moved one position to the right, and thus these numbers have been moved $i - 1$ places to the right total. Thus in the current tableau when we are moving n_i into position, $a_l = A_l + i - 1 = l + i - 1 - m + A_m$. Thus, our product becomes

$$\prod_{l=m}^{n_i} \frac{(l+i-1-m+A_m-A_{c_i})^2}{(l+i-2-m+A_m-A_{c_i})(l+i-m+A_m-A_{c_i})}.$$

This product telescopes to leave a positive constant times

$$\frac{n-d_i+i-1-m+A_m-A_{c_i}}{n-d_i+i-m+A_m-A_{c_i}}.$$

Multiplying over i implies the desired formula for the overall norm. $\square$

4 Signatures

This norm formula tells us that the signature is always positive or negative definite in many cases, and gives an algorithm for computing it in all other cases. We have calculated these signatures for many cases, and in this section we give an example of the particularly simple and elegant case of hooks, followed by a theorem that simplifies signatures computation in many other cases.

4.1 Signatures for Hooks

We consider restrictions from a larger hook to a smaller hook, so the added boxes consist of a section of the first row and a section of the first column. The formulas in this case are particularly simple and elegant, and lead to the natural generalization of a single column plus an arbitrary configuration of separate squares, which we will explore in the next section.

Theorem 7. *In the case of a single hook where j boxes are added to the first row and $k-j$ to the first column, the norm is $\binom{k}{j} - 2\binom{k-1}{j}$ for $0 < n < k$, $n \notin \mathbb{N}$, and $\binom{k}{j}$ otherwise.*

Proof. Our norm formula gives us $\prod_{i=1}^{k-j} \frac{n - d_i + i - 1 - m + A_m - A_{c_i}}{n - d_i + i - m + A_m - A_{c_i}}$, where $n - d_{k-j}, ..., n - d_1$ are the entries in the first column for some $0 \leq d_1 < d_2 < ... < d_{k-j} \leq k-1$.

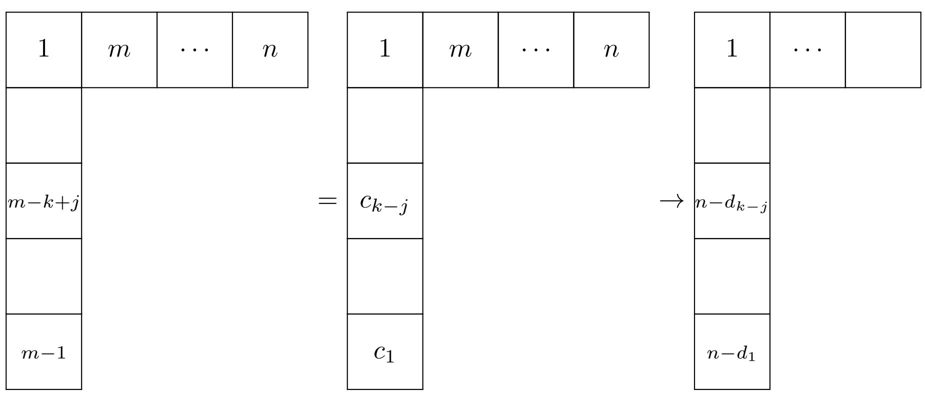


Figure 10: Corresponding standard column tableau followed by a basis vector.

Let m be the second entry in the first row of the standard Young tableau of our shape, so $A_m = 2-1 = 1$ (Fig. 10). Then the last $k-j$ entries in the first column (i.e. the $k-j$ added boxes in that column) have entries $m-k+j, m-k+j+1, ..., m-1$. These are the c_i values, and they must be ordered from greatest to least since c_1 corresponds to the square where the largest value ends up, c_2 where the second largest value ends up, and so on, and all the c_i are in the same column. Thus, $c_i = m - i$. Also, $A_{c_i} = 1 - c_i = 1 - m + i$. Substituting these values into our numerator gives $n - d_i + i - 1 - m + A_m - A_{c_i} = n - d_i + i - 1 - m + 1 - 1 + m - i = n - d_i - 1$.

The denominators of the fractions in the product are 1 more than the numerators, so the product becomes $\prod_{i=1}^{k-j} \frac{n - d_i - 1}{n - d_i}$. Since the d_i are all distinct, this product is negative if and only if $d_i < n < d_i + 1$ for some

d_i. Since $0 \leq d_i \leq k-1$ for all i, this is impossible for $n < 0$ or $n > k$, so the norm is positive definite in these cases.

Choosing a basis vector is equivalent to choosing which $k-j$ of the k numbers $n-k+1, n-k+2, ..., n$ are in the first column, since the numbers must be increasing in both the first row and first column, so their ordering is fixed, thus the total number of basis vectors is $\binom{k}{j}$.

If $0 < n < k$, $n \notin \mathbb{N}$, then there is exactly one positive integer $0 \leq d \leq k-1$ such that $d < n < d+1$, and the norm will be negative iff this d is one of the d_i values. A basis vector is entirely determined by the choice of d_i values, and they must all be distinct, so choosing a basis vector where d is one of the values is equivalent to choosing the remaining $k-j-1$ values used from the remaining $k-1$ possible values. This can be done in $\binom{k-1}{k-j-1} = \binom{k-1}{j}$ ways, so this is the number of basis vectors with negative norm, and the remaining $\binom{k}{j} - \binom{k-1}{j}$ have positive norm, giving a signature of $\binom{k}{j} - 2\binom{k-1}{j}$.

□

4.2 One Term per Column Theorem

In this section, we prove that the root pairs from a given column are all distinct, so that only one square per column can give a negative contribution to the norm. This greatly reduces the complexity of signature computations by limiting the number of negative factors to the number of columns.

Theorem 8. *At most one term from a square in any column can be negative.*

Proof. By Theorem 6, a pair of roots has a nontrivial contribution (the numerator is negative and the denominator positive), iff $\lfloor n \rfloor = d_i - i + m - A_m + A_{c_i}$ for some i such that $n - d_i$ is in a box in our column. Suppose this were true for multiple values of i. We know that the sequence $d_1, d_2, ..., d_{k-j}$ is strictly increasing.

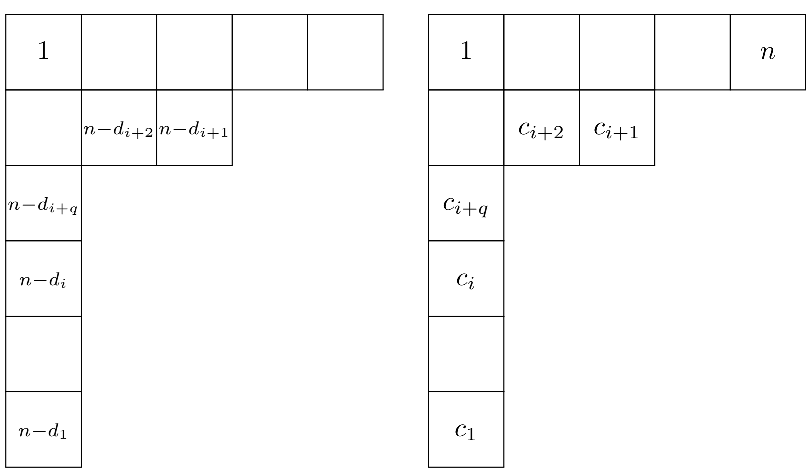


Figure 11: (a) A basis vector tableau; (b) the corresponding column tableau

If $n - d_i$ is in our column, then $n - d_{i+q}$ is directly above it for some $q \geq 1$, since the lower square was filled first and thus the higher square has a smaller entry and a larger i value (Fig. 11a). Then the contents of c_{i+q} and c_i will differ by 1, so $A_{c_{i+q}} = A_{c_i} + 1$ (Fig. 11b), and we know that $d_{i+q} \geq d_i + q$, thus $d_{i+q} - (i+q) + m - A_m + A_{c_{i+q}} \geq d_i + q - i - q + m - A_m + A_{c_i} + 1 > d_i - i + m - A_m + A_{c_i}$. Therefore, this value $d_i - i + m - A_m + A_{c_i}$ is strictly increasing moving up our column and thus cannot take the same value twice. Therefore, at most one of the squares in our column gives a nontrivial contribution to the norm for its corresponding terms. $\square$

Suppose p squares are not in our column or the first row, and let c be the lowest entry in our column in the column tableau. Let N be the number of orderings of the numbers in the remaining p squares. By the hook length formula, N is $p!$ divided by the lengths of all hooks formed by any of the p squares [12].

Lemma 9. *The total number of basis vectors is* $\binom{k}{p} N \binom{k-p}{j}$.

Proof. Of the k numbers $n-k+1, n-k+2, ..., n$, p must be in the squares not in our column, and these p can be chosen in $\binom{k}{p}$ ways. No matter how they are chosen, they are all distinct, and there are exactly N ways to order them. There are then $\binom{k-p}{j}$ ways to choose the j numbers in the first row, and the remaining $k-p-j$ numbers are in our column.

The numbers in the first row must increase from left to right, and the numbers in our column must increase from top to bottom, so there is only one basis vector with these properties. We are assuming our column does not contain squares adjacent to the other p squares, so the choice and ordering of these numbers do not affect the ordering of numbers in the other p squares. Thus, the total number of basis vectors is $\binom{k}{p} N \binom{k-p}{j}$. $\square$

Lemma 10. *Our column does not change the signature unless*

$$n \in (m - A_m + A_c - 1, m - A_m + A_c + k - p).$$

Proof. We know that a basis vector has negative norm iff $\lfloor n \rfloor = d_i - i + m - A_m + A_{c_i}$ for some i such that $n - d_i$ is in our column. We also know that these numbers are strictly increasing moving up the column. Thus, the minimum value this can take is for the lowest square, when $d_i - i = -1$, giving $m - A_m + A_c - 1$. The maximum value is for the highest square, where $A_{c_i} = A_c + k - j - p - 1$, since this is the number of squares in this column, and $d_i - i$ takes on its maximum value of j. Thus, $d_i - i + m - A_m + A_{c_i} \leq m - A_m + A_c + k - p - 1$, so the only way the norm can be negative is if $n \in (m - A_m + A_c - 1, m - A_m + A_c + k - p)$. $\square$

4.3 Signature Contribution from One Column

In this section, we find a formula for the number of basis vectors with a negative term from a given column. This immediately leads to signature formulas if the columns are sufficiently far apart. Consider a single column of q added squares:

Theorem 11. *The number of basis vectors with a negative term from a column of added squares of length q, with j added squares in the first row, is given by*

$$\frac{q \dim V_{\lambda,\rho}}{q+j}.$$

Proof. Since a basis vector is formed by choosing the $j+q$ numbers either in our column or the first row, then choosing the j of these $k+j$ numbers that are in the first row, and then ordering the remaining squares, the dimension of $V_{\lambda,\rho}$ is

$$\binom{k}{j+q}\binom{j+q}{j}N$$

for a constant N that counts the number of ways to order the entries in the remaining $k-j-q$ squares (the orderings for our column and the first row are determined since the tableau is standard). The only way a square in the column can give a negative factor is if for some I

$$\lfloor n \rfloor = d_I - I + m - A_m - A_{c_I} \iff d_I - I + A_{c_I} = \lfloor n \rfloor - m + A_m.$$

Note that there are a total of $k-j$ of the $n-d_i$ values in added squares not in the first row, $k-j-q$ of which are not in our column, and that the d_i increase strictly with i and range from 0 to $k-1$. Let c be the bottom square in our column, and suppose there are $0 \leq l \leq k-j-q-1$ of the $n-d_i$ not in our column with $i < I$, so $n-d_I$ is in the $(I-l)^{th}$ square from the bottom of the column. Then

$$d_I - I + A_{c_I} = d_I - I + (A_c + I - l - 1) = \lfloor n \rfloor - m + A_m \iff d_I - l = \lfloor n \rfloor - m + A_m - A_c + 1.$$

Call the RHS x. To form a basis vector with $d_I = l + x$, we must choose the l d_i values that are less than d_I and not in our column from the $x+l$ options less than $d_I = x+l$, then choose the remaining $k-j-q-l$ of the d_i values in the squares not in our column from the $k-x-l-1$ choices that are larger than $x+l$. We then choose the j first row numbers from the remaining $j+q-1$ options, and order them in N possible ways, as before. The number of basis vectors with a negative contribution from our column is

$$\left(\sum_{l=0}^{k-j-q}\binom{x+l}{l}\binom{k-x-l-1}{k-j-q-l}\right)\binom{j+q-1}{j}N = \binom{k}{k-j-q}\binom{j+q-1}{j}N = \frac{q \dim V_{\lambda,\rho}}{q+j}.$$

If the other columns have contents sufficiently far from the ones in our column, then for n in a range where one of the d_i values can equal $x+l$ for each l, the signature is

$$\frac{j \dim V_{\lambda,\rho}}{q+j}.$$

$\square$

5 Conclusions

5.1 Discussion

The representation theory of the symmetric groups S_n is fairly well understood. The irreducible representations of S_n have been characterized using Young diagrams, and their dimensions and characters, as well as the decompositions of induced and restricted representations, are known. The representations of S_n are also known to have deep connections to algebra and combinatorics.

The representation theory of finite groups extended to arbitrary real or complex rank, however, is relatively unexplored, and this is an exciting new field of study. A construction of Deligne categories Rep S_t has been given, and some formulas can easily be interpolated from the positive integer case. However, much is unknown about the structures of these categories, and these structures are often much more interesting and insightful than in the degenerate positive integer cases.

In this project, we studied signatures, an important invariant property of Deligne categories, in the case of multiplicity spaces $V_{\lambda,\rho}$. We have found explicit formulas for the norms of an orthogonal basis of this space under the invariant inner product, which give an algorithm for computing signatures for any λ and ρ, and we have also found useful properties of the signatures, as well as formulas in certain cases.

We have made a number of interesting observations from our norm and signature formulas:

- The norms are rational functions of n with positive integer roots.
- The roots some in pairs that differ by one, with a root $r-1$ in the denominator for each root r in the numerator.
- Each pair of roots corresponds to an added square not in the first row.
- The norm is always positive definite for sufficiently large or small n.
- The norm is always positive or negative definite when no added square are in the first row of the Young diagram.
- Root pairs for the same column are all distinct, so for a given n at most one term from a given column can be negative.
- The fraction of basis vectors with a negative term from a given column depends only on the length of the column and number of added squares in the first row.

We had expected the norms to be polynomials or rational functions of n, but the fact that the roots come in pairs was surprising and has very interesting implications for the signatures, such as showing that they are positive definite for not only very large but also very small n. It is also interesting that the signature is positive or negative definite in a surprisingly large number of cases (whenever λ and ρ have the same first element), and that signatures are so closely connected to columns in Young diagrams. These results also give an idea of what signatures may be like in other cases, and it will be interesting to see how many of these properties are shared by other signatures in Deligne categories.

5.2 Future Research

In addition to further studying these norm formulas and the resulting signatures, an interesting and closely related problem would be to compute norms and signatures for spaces counting the multiplicity of the tensor product $V_\lambda \otimes V_\mu$ in the restriction of the irreducible V_ν of $S_{|\nu|}$ to the product $S_{|\lambda|} \otimes S_{|\mu|}$, where $|\nu| = |\lambda| + |\mu|$.

The dimensions of these multiplicity spaces are given by Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$, which have a number of combinatorial and algebraic descriptions, such as the number of semistandard skew tableaux of shape ν/λ and weight μ such that the sequence formed by concatenating its reversed rows is a lattice word, as well as the coefficient of the Schur polynomial s_ν in the product $s_\lambda s_\mu$ written with respect to the basis of Schur functions for the ring of symmetric functions. Signatures in this case will naturally be closely tied to Littlewood-Richardson coefficients and will thus likely have important combinatorial implications, as well as interesting connections to the case we have studied, so this would be a natural direction for future study.

Another related but more difficult problem would be studying norms and signatures for the multiplicity spaces of V_ν in the tensor product $V_\lambda \otimes V_\mu$. The dimensions of these spaces are given by Kronecker coefficients $g^{\nu}_{\lambda\mu}$, which still do not have a known combinatorial description (another major unsolved problem in the representation theory of the symmetric group), and this method for studying them could potentially shed some light on that.

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