The Defect Angle and the Relation to the Laplacian Matrix

Brandon Rafal Epstein

Mentor: Dr. Martin Roček
Finite Triangulation

- Tiling of any two-dimensional surface with triangles
- Pillow triangulation of a sphere

Author, Brandon Rafal Epstein
Finite Triangulation

- Approximation of smooth surface improves as number of triangles increases

Source: http://ieeexplore.ieee.org.ieee_pilot/articles/05/ttg2009050719/assets/img/article_1/fig_3/large.gif
A measure of the angle "missing" from a vertex of the triangulation

\[ \epsilon_i = 2\pi - \sum_{(j,k)| (i,j,k) \in F} (\alpha_{jik}) \]
Demonstration of angular deficit and surplus

Source: http://royalsocietypublishing.org/content/royprsa/469/2153/20120631/F1.large.jpg
Defect angles in a regular tetrahedron. The defect angle at each vertex is $\pi$ because there are 3 angles measuring $\frac{\pi}{3}$. 
What is $\Gamma$?

- A function on a finite triangulation, determined by A. Ko and M. Roček, equal to:

$$\Gamma = \frac{1}{12\pi} \left[ \sum_{\triangle ijk} \left( \alpha_{ijk} \int_{\pi/2}^{y} \left( y - \frac{\pi}{3} \right) \cot y \, dy \right) + \sum_{\langle ij \rangle} 2k_{ij}\pi \ln \left( \frac{\ell_{ij}}{\ell_{0}} \right) \right]$$

- Constants $k_{ij}$ are defined such that

$$\sum_{j \mid \langle ij \rangle \in E} k_{ij} = 1 - \frac{n_{i}}{6}$$

where $n_{i}$ is the number of vertices adjacent to $i$.

- Defined so that if we change each edge $\ell_{ij}$ at $i$ by $\alpha_{i}\ell_{ij}$, then $\Gamma$ will change proportionally to $\alpha_{i}\epsilon_{i}$. 

Author, Brandon Rafal Epstein

The Defect Angle
Rescaling a vertex

The effect of rescaling \( \langle 1, 2 \rangle \) on the triangulation. Only two of the six triangles are affected, showing the locality of the rescaling and \( \Gamma \).

Author, Brandon Rafal Epstein  The Defect Angle
Phi Quantities ($\Phi_i$)

- Consider a triangulation where the edge lengths are given by

\[ \ell_{ij} = \ell_{ij}^0 e^{\Phi_i + \Phi_j} \]

- We can rescale the triangulation’s edge lengths by adding constants to $\Phi_i$ and $\Phi_j$.
- $\frac{\partial \Gamma}{\partial \Phi_i}$ is proportional to $\epsilon_i$. 

Author, Brandon Rafal Epstein

The Defect Angle
The principal problem we are investigating:

- What relations can we find between the properties of discrete triangulations and those of smooth surfaces?

An interesting question we explored in passing:

- What is the Taylor series of $\Gamma$, and what information about a triangulation is conveyed in its coefficients?
Methodology and Procedure

- **Methodology**
  - Multivariate calculus

- **Procedure**
  - Second-order Taylor series with respect to the $\Phi$ values
  - Laplace operator and Laplace matrix
The energy functional is given by an integral involving the Laplace operator $\nabla^2 \Psi$ on an arbitrary function $\Psi$:

$$- \int \int_D \Psi \nabla^2 \Psi \, dx \, dy$$

Integrating by parts we can rewrite this as:

$$\int \int_D (\nabla \Psi \cdot \nabla \Psi) \, dx \, dy$$
Laplacian matrix \((L)\)

- Discrete analogue of the Laplace operator
- Acts on a matrix \(\Phi\) by matrix multiplication \((L \cdot \Phi)\)
- Xianfeng Gu et al. define a modified Laplace matrix:

\[
L_{ij} = \begin{cases} 
-w_{ij} & \text{if } i \neq j \\
\sum_k w_{ik} & \text{if } i = j
\end{cases},
\]

\[
w_{ij} = \begin{cases} 
\frac{\cot \alpha_{ikj}}{2} & \text{if } [v_i, v_j] \in \partial M \\
\sum_{k \mid (i,j,k) \in F} \frac{\cot \alpha_{ikj}}{2} & \text{if } [v_i, v_j] \notin \partial M
\end{cases}
\]

- The \(w_{ij}\) terms are named **cotangent edge weights**.
Cotangent edge weights

Interior edge \( \langle ij \rangle \) has

\[
    w_{ij} = \frac{1}{2} (\cot \alpha_{ik_1j} + \cot \alpha_{ik_2j})
\]
so \( L_{ij} = -\frac{1}{2} (\cot \alpha_{ik_1j} + \cot \alpha_{ik_2j}) \)

Boundary edge \( \langle jk_1 \rangle \) has

\[
    w_{jk_1} = \frac{1}{2} \cot \alpha_{jik_1}
\]
so \( L_{jk_1} = -\frac{1}{2} \cot \alpha_{jik_1} \)

\[
    L_{k_1k_1} = w_{ik_1} + w_{jk_1} = \frac{1}{2} (\cot \alpha_{jik_1} + \cot \alpha_{ijk_1})
\]

As \( k_1 \) and \( k_2 \) are not adjacent,
\( L_{k_1k_2} = 0. \)
We need to compute the first and second derivatives of $\Gamma$ with respect to the $\Phi$ quantities.

By construction, the first derivative of $\Gamma$ with respect to $\Phi_i$ is:

$$\frac{\partial \Gamma}{\partial \Phi_i} = \epsilon_i$$
• There are two cases of second derivatives of $\Gamma$ with respect to $\Phi$:
  
  • "On-diagonal": the second derivatives of the form
    
    \[
    \frac{\partial^2 \Gamma}{\partial \Phi_i^2}
    \]
  
  • "Off-diagonal": the second derivatives of the form
    
    \[
    \frac{\partial^2 \Gamma}{\partial \Phi_i \cdot \partial \Phi_j}
    \]

• We use the defect angle to compute these derivatives.
• We start by differentiating an angle of a triangle, $\alpha_{jik}$.
• We use the Law of Cosines:

\[
\cos \alpha_{jik} = \frac{\ell_{ij}^2 e^{2\Phi_i+2\Phi_j} + \ell_{ik}^2 e^{2\Phi_i+2\Phi_j} - \ell_{jk}^2 e^{2\Phi_j+2\Phi_k}}{2\ell_{ij} e^{\Phi_i+\Phi_j} \ell_{ik} e^{\Phi_i+\Phi_k}}
\]

• By differentiating both sides with respect to a $\Phi$ value, we can isolate the derivative of the angle.
To recapitulate, we have the following derivatives:

\[
\frac{\partial \Gamma}{\partial \Phi_i} = \epsilon_i
\]

\[
\frac{\partial^2 \Gamma}{\partial \Phi_i^2} = - \sum_{i | (i,j,k) \in F} \cot \alpha_{ikj} + \cot \alpha_{ijk}
\]

\[
\frac{\partial^2 \Gamma}{\partial \Phi_i \cdot \partial \Phi_j} = - \sum_{i,j | (i,j,k) \in F} \cot \alpha_{ikj}
\]

Substituting these into the general formula of a multivariable Taylor series, we finish the derivation.
The Taylor series of $\Gamma$ to the second order is calculated to be:

$$
\Gamma = \Gamma_0 + \frac{1}{12\pi} \left( \sum_{i \in V} (\epsilon_i \Phi_i) \right) 
+ \sum_{i \in V} \left( \sum_{j,k | (i,j,k) \in F} \left( \frac{\cot \alpha_{ijk} + \cot \alpha_{ikj}}{2} \right) \Phi_i^2 \right) 
+ \sum_{\langle ij \rangle \in E} \left( \sum_{k | (i,j,k) \in F} \left( -\cot \alpha_{ikj} \Phi_i \Phi_j \right) \right)
$$

This series had not been calculated previously.
M. Roček and R. M. Williams calculated the previous integral
\[ \int \int_D (\nabla \Phi \cdot \nabla \Phi) \, dx \, dy \]

They determined that this is equal to:
\[
\frac{1}{2} \left( \left( \frac{\cot \alpha_2 + \cot \alpha_3}{2} \right) \Phi_1^2 + \left( \frac{\cot \alpha_1 + \cot \alpha_3}{2} \right) \Phi_2^2 + \left( \frac{\cot \alpha_1 + \cot \alpha_2}{2} \right) \Phi_3^2 \right) + \left( - \cot \alpha_3 \right) \Phi_1 \Phi_2 + \left( - \cot \alpha_2 \right) \Phi_1 \Phi_3 + \left( - \cot \alpha_1 \right) \Phi_2 \Phi_3 \]

which is proportional to the second-order terms of the Taylor series of $\Gamma$ for a pillow triangulation by a factor of $6\pi$. 
Results

- The three quantities we have discussed are all equal!
  - The coefficients of second-order terms of the Taylor series of $\Gamma$
  - The entries of the Laplace matrix
  - The coefficients of the expansion of the energy functional
In general, further research topics are those which discretize other continuous concepts.

- Cauchy-Riemann Equation - useful tool in the continuous case; holomorphic criteria for complex functions. Holomorphic functions give solutions to the Liouville equation.
- Liouville theory - concerns solutions to the Liouville equation in the continuous case.
We determine the Taylor series expansion of $\Gamma$ with respect to $\Phi$ quantities.

We verify that the Taylor series expansion, the gradient integral, and the Laplace matrix are (up to proportionality factors) equivalent.
I wish to thank:

- Dr. Martin Roček, my mentor
- Dr. Tanya Khovanova
- The PRIMES program, Prof. Pavel Etingof, and Dr. Slava Gerovitch
- Anton Wu
- Dr. Michael Lake and Mrs. Maria Archdeacon
- My mother, father, and grandmother
References