Equal Compositions of Rational Functions

Kenz Kallal, Matthew Lipman, Felix Wang
Mentors: Thao Do and Professor Michael Zieve

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## The Problems

A rational function is a ratio of two polynomials.

<table>
<thead>
<tr>
<th>Problem 1</th>
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<tbody>
<tr>
<td>Find all rational functions $a, c \in \mathbb{Q}(X)$ such that $a(Y) = c(Z)$ has infinitely many solutions for $Y, Z \in \mathbb{Q}$.</td>
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One source of solutions to Problem 1 comes from the following problem when the functions have rational coefficients:

<table>
<thead>
<tr>
<th>Problem 2</th>
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<td>Find all rational functions $a, b, c, d \in \mathbb{C}(X)$ such that $a(b(X)) = c(d(X))$.</td>
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SOME EXAMPLES:

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X^m \circ X^n = X^n \circ X^m = X^{mn}
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- For an arbitrary rational function $h(X)$,
  $X^2 \circ Xh(X^2) = Xh(X)^2 \circ X^2 = X^2h(X^2)^2$. 
**RESULT**

Theorem

*If the numerator of $a(X) - c(Y)$ is irreducible, then one of the following must hold:*

- $\deg a, \deg c \leq 250$
**Result**

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- $\deg a, \deg c \leq 250$
- at least one of $a$ and $c$ are “nice” functions (e.g. $X^m$, Chebyshev, functions coming from elliptic curves)
- Up to change in variables,

$$a = X^i(X - 1)^j, c = rX^i(X - 1)^j.$$
OUTLINE OF OUR STRATEGY

Rational function problems

Ramification multisets of $a$ and $c$

Rational function problems

Faltings’, Riemann-Hurwitz Formula

Ramification Multiset Conditions

Hurwitz’s Theorem, reducibility checking

Combinatorics, computer programs
RAMIFICATION

Definition (Ramification)

- The ramification index $e_f(P)$ of $f$ at a point $P$ is the multiplicity of $P$ as a root of $f(X) - f(P)$. 
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- Example: \( f(X) = X^3 + X^4 = X^3(X + 1) \) has \( E_f(0) = [3, 1] \).
**MULTISET PROBLEM**

The multiset problem

If the numerator of $a(X) - c(Y)$ is irreducible,

N.1. $\sum_{i \in A_k} i = m$ and $\sum_{i \in C_k} i = n$ for each $k$ ($m$ and $n$ are the degrees of $a$ and $c$ and $A_k$ and $C_k$ are ramification multisets of $a$ and $c$).

N.2. $\sum_{k=1}^{r} (m - |A_k|) = 2m - 2$ and $\sum_{k=1}^{r} (n - |C_k|) = 2n - 2$.

N.3. $\sum_{k=1}^{r} \sum_{i \in A_k} \sum_{j \in C_k} (i - \gcd(i,j)) \in \{2m - 2, 2m\}$. 
SOLVING THE MULTISET PROBLEM

Let $m, n$ denote the degrees of $a$ and $c$. We will assume that $n \geq m$. We split into 3 cases:

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- Globally: We find all the possibilities for $\{k_i\}$.
- For each possibility of $\{k_i\}$, we solve for $\{A_i\}$.
**Results**

**Proposition**

If rational functions $a$ and $c$ are solutions to the multiset problem, then at least one of $a$ and $c$ satisfies

$$\sum_{k=1}^r \left(1 - \frac{1}{\text{lcm}(F_k)}\right) \leq 2$$

where $\{F_k\}$ is the list of all ramification multisets of that function.
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SOLVING FOR THE $A_i$

1. $A_1 \cup A_2 \cup A_3 \cup A_4 = [1^4, 2^{2m-2}]$.

8. $A_1 = A_2 = [m]$. 
**Solving for the $C_i$**

For example, suppose that $A_1 = A_2 = [m]$. This corresponds to $a(X) = X^m$.

1. $c(X) = h(X)^m X^k$ for $k$ relatively prime to $m$,
2. $m = 6$ and $c(X) = h(X)^6 X^3 (X - 1)^{\pm 2}$,
3. $m = 4$ and $c(X) = h(X)^4 X^2 (X - 1)^{\pm 1}$,
4. $m = 3$ and $c(X) = h(X)^3 X^{\pm 1} (X - 1)^{\pm 1}$ (with the $\pm$ independent),
5. $m = 2$ and $c(X)) = h(X)^2 X (X - 1) (X - X_0)$ (with $0 \neq x_0 \neq 1$),

where $h(X)$ is any rational function.
BACK TO THE ORIGINAL PROBLEMS

- checking that functions $a$ and $c$ exist.
Back to the original problems

- checking that functions $a$ and $c$ exist.
- determining whether $a(X) - c(Y)$ is irreducible
EXISTENCE OF RATIONAL FUNCTIONS

Hurwitz’s Theorem

A finite collection of $k$ multisets $A_i$ of sum $n$ with corresponds to a rational function if and only if both of the following are true:

\[ \sum_{i} \leq k \left( n - |A_i| \right) = 2n - 2. \]

There exist permutations $g_1, \ldots, g_k \in S_n$ such that $g_i$ has cycle structure $A_i$ and the product of the permutations is the identity. Furthermore, the group generated by $g_1, \ldots, g_k$ must be transitive.
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Testing for irreducibility

Extra Condition

For all $i, j \leq r$, $A_i \cup A_j \cup C_i \cup C_j$ has greatest common divisor equal to one.
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Theorem (Reducibility test)

If $\sum_{k=1}^{r} \sum_{i \in A_k} \sum_{j \in C_k} (i - \gcd(i, j)) < 2m - 2$, any rationals $a(X)$ with multisets $A_k$ and $c(Y)$ with multisets $C_k$ will have $a(X) - c(Y)$ reducible.

This is similar to one of our previous conditions, so we usually keep $c$ the same and vary $a$ to show that $c$ is decomposable so that $a(X) - c(Y)$ is reducible.
**Future research**

- Finish finding $a$ and $c$ for the case in which $a$’s multisets have small lcm.
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FUTURE RESEARCH

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- Continue to lower the bounds for 250 and 10 above.

- The case in which \(a(X) - c(Y)\) is not irreducible.
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