XYX Algebras

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Algebras Generated by Graphs

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**Algebras Generated by Graphs**

- Given a graph $G$, we construct an algebra associated with it.
- Each vertex represents a generator.
- Generators follow certain relations based on whether the corresponding vertices are adjacent.
XYX Algebras

In an XYX algebra, the following relations hold:

- If two generators, $x$ and $y$ are not adjacent, then they commute ($xy = yx$).
- If two generators, $x$ and $y$ are adjacent, then $xyx = 0$ and $yxy = 0$.
- The square of any generator is zero ($xx = 0$).
FINITE VS. INFINITE ALGEBRAS
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**Conjecture**

The only graphs corresponding to finite XYX algebras are the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, and $E_8$.

![Dynkin diagrams]
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**Conjecture**

*The only graphs corresponding to finite XYX algebras are the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$, and $E_8.*

This suggests a natural connection between Coxeter groups and XYX algebras.
Examples of Graphs with Infinite Algebras

We have identified several conditions on the graph that make the algebra infinite:

- Cycle
- Vertex of degree 4
- Two or more vertices of degree 3
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This limits all graphs with a finite algebra to either the path graphs or graphs with one vertex that has three paths branching out of it.
**Cycle Graphs**

- If we have a cycle made with generators $x_1, x_2, ..., x_n$, then we can have an infinite number of words of the form $x_1x_2...x_nx_1.....$
**Vertex of Degree 4**

- If we have a vertex of degree 4, then we can form an infinite number of words by repeating

  \[x_1 x_2 x_3 x_1 x_4 x_5 x_1 x_2 x_3 x_1 x_4 x_5\]
If we have two or more vertices of degree 3, then we can form an infinite number of words by repeating

\[ abx_1x_2x_3x_4cdx_4x_3x_2x_1abx_1x_2x_3x_4cdx_4x_3x_2x_1 \]
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![Path graph diagram]
First, we calculated the dimension of the algebra for small $n$. The dimensions were 1, 2, 5, 14, and 42, which are Catalan numbers. Theorem. The dimensions of the XYX algebras of the path graphs are the Catalan numbers.
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Theorem

*The dimensions of the XYX algebras of the path graphs are the Catalan numbers.*
Catalan Numbers

- One of the most common sequences in mathematics
- Over 150 definitions
- One such definition is the Dyck paths
Dyck Paths

- Dyck paths are a way of representing the Catalan numbers as a set of paths between two points.
- Below is a sample Dyck path of length 8.
\[ A_n \]

- We counted the words in the algebra by length.
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We discovered an interesting connection to Dyck paths.

Theorem
The number of words of a given length in the XYX algebra of a path graph is the same as the number of Dyck paths where the sum of the heights of the peaks minus the number of peaks is the length of the word.
We wanted to find a simple bijection between words in the algebra and Dyck paths.
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We build the bijection as follows.
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$A_n$
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- Write each run as a pair $(p_i, r_i)$, where $p_i$ is the first generator of the run and $r_i$ is the length of the run.
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Write each run as a pair \((p_i, r_i)\), where \(p_i\) is the first generator of the run and \(r_i\) is the length of the run.
Draw a mountain top with peak coordinates \((p_i + r_i + 1, r_i + 1)\).
$A_n$

- Divide the word into descending runs of generators.
- Write each run as a pair $(p_i, r_i)$, where $p_i$ is the first generator of the run and $r_i$ is the length of the run.
- Draw a mountain top with peak coordinates $(p_i + r_i + 1, r_i + 1)$.
- Superimpose all the mountains.
- Add trivial peaks if necessary.
EXAMPLES

- Here is the sample Dyck path corresponding to $x_2x_1x_3x_2$.
- The two peaks correspond to the descending runs $x_2x_1$ and $x_3x_2$. 
There are many future directions we could take this project.

- Finish our proof that only the Dynkin diagrams of finite Coxeter groups produce a finite XYX algebra
- Enumerate the dimensions on algebras corresponding to other graphs, such as $D_n$
- Enumerate the number of words of a given length in an infinite algebra
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