Analysis of Boolean Functions

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Boolean Functions

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Applications of Boolean functions:
- Circuit design.
- Learning theory.
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Applications of Boolean functions:

- Circuit design.
- Learning theory.
- Voting rule for election with \( n \) voters and 2 candidates \( \{-1, 1\} \); social choice theory.
Convention: $x \in \{-1, 1\}^n$; $x_1, x_2, \ldots, x_n$ are coordinates of $x$. 

Majority function: $\text{Maj}_n(x) = \text{sgn}(x_1 + x_2 + \cdots + x_n)$. 

- $(1, 1, 1)$ 
- $(1, -1, -1)$ 
- $(-1, 1, -1)$ 
- $(-1, -1, 1)$ 

$f$ is linear threshold function (weighted majority) if $f(x) = \text{sgn}(a_0 + a_1 x_1 + \cdots + a_n x_n)$. 

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Boolean functions
Majority, Linear Threshold Functions

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![Diagram of Majority Function](image)
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$$
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$$
AND, OR, Tribes

- $-1 \leftrightarrow \text{True}, 1 \leftrightarrow \text{False}$.
- $\text{AND}_n(x) = x_1 \land x_2 \land \cdots \land x_n$.
- $\text{OR}_n(x) = x_1 \lor x_2 \lor \cdots \lor x_n$. 

$\text{Tribe}_{w,s}(x_1, \ldots, x_{sw}) = (x_1 \land \cdots \land x_w) \lor \cdots \lor (x_{(s-1)w} \land \cdots \land x_{sw})$.

$n = ws$ is the number of voters. $s$ tribes, $w$ people per tribe.
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  - \( s \) tribes, \( w \) people per tribe.
### Influence

#### Definition

**Impartial culture assumption:** $n$ votes independent, uniformly random: 
$x \sim \{-1, 1\}^n$.

Influence at coordinate $i$, $\text{Inf}_i[f]$: prob. that voter $i$ changes outcome.

Influence of $f$: 
$I[f] = \sum_{i=1}^{n} \text{Inf}_i[f]$.

Example: 
$I[\text{Maj}_3(x)] = \frac{3}{2}$.
Influence

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- **Example:** $\mathbf{I}[\text{Maj}_3(x)] = \frac{3}{2}$.

Example:

\[
\begin{pmatrix}
1, 1, 1 \\
-1, -1, 1 \\
-1, -1, -1
\end{pmatrix}
\]
Nassau County (NY) voting system:

\[ f(x) = \text{sgn}(-58 + 31x_1 + 31x_2 + 28x_3 + 21x_4 + 2x_5 + 2x_6). \]
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Lawyer Banzhaf sued Nassau County board (1965).
Influences of Tribes, Majority

\[ f \text{ monotone: } x \leq y \text{ coordinate-wise } \Rightarrow f(x) \leq f(y). \]

**Theorem**

\[ I[f] \leq I[M_{\text{maj}}] = \sqrt{2/\pi} \sqrt{n} + O(n^{-1/2}) \text{ for all monotone } f. \]
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- For \( n = ws \), define \( \text{Tribes}_n = \text{Tribes}_{w,s} \) with \( w, s \) such that \( \text{Tribes}_{w,s} \) is essentially unbiased.
- \( \text{Inf}_i[\text{Tribes}_n] = \frac{\ln n}{n} \cdot (1 + o(1)). \)
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**Theorem (Kahn, Kalai, Linial)**

\[
\text{MaxInf}[f] \geq \text{Var}[f] \cdot \Omega\left(\frac{\log n}{n}\right).
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- Application: bribing voters.
Another important property of Boolean functions: noise stability.

**Definition**

For a fixed $x \in \{-1, 1\}^n$ and $\rho \in [0, 1]$, $y$ is $\rho$-correlated with $x$ if, for each coordinate $i$, $y_i = x_i$ with probability $\rho$ and randomly chosen with probability $1 - \rho$.

**Definition**

For a Boolean function $f$ and $\rho \in [0, 1]$, the noise stability of $f$ at $\rho$ is $\text{Stab}_\rho[f] = E[f(x)f(y)]$ for $x$ uniformly random and $y$ $\rho$-correlated with $x$.
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Noise Stability

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Noise Stability: Voting Example
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- Voting system represented by $f$.
- Some chance $\frac{1-\rho}{2}$ that the vote is misrecorded.
- Noise stability: measure of how much $f$ is resistant to misrecorded votes.
The Noise Stability of Majority

Natural question: what is the noise stability of Majority?

Theorem
For any $\rho \in [0, 1]$, \[ \lim_{n \to \infty} \text{Stab}_\rho [\text{Maj}_n] = 2\pi \arcsin \rho. \]

General idea of proof: use the multidimensional central limit theorem.

Theorem (Majority is Stablest)
Among Boolean functions that are unbiased and have only small influences, the Majority function has approximately the largest noise stability.
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**Theorem (Majority is Stablest)**

*Among Boolean functions that are unbiased and have only small influences, the Majority function has approximately the largest noise stability.*
Noise stability also key in proving Arrow’s Theorem. In particular, consider:

Two candidate elections: most fair voting rule is Majority.

Three candidate elections: not clear how to conduct the election.

One possibility: conduct pairwise Condorcet elections, each of which is evaluated by some voting rule $f$. The Condorcet winner is the candidate that wins all his/her elections. May not always occur: might be some situations in which each candidate loses a pairwise election.

Goal: find a function in which this contradiction never occurs.
Arrow’s Theorem: The Idea

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Example of contradiction: candidates $A, B, C$; voters $x_1, x_2, x_3$; voting rule is Majority function.
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$A$ wins the pairwise election with $B$.
$C$ wins the pairwise election with $A$.
$B$ wins the pairwise election with $C$.
There is no Condorcet winner!
Example: Contradiction with $f$ Majority

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Arrow’s Theorem: The Statement

**Theorem (Arrow’s Theorem)**

*In an n-candidate Condorcet election, if there is always a Condorcet winner, then* \( f(x) = \pm x_i \) *for some* \( i \) *(dictatorship).*
Theorem (Arrow's Theorem)

In an $n$-candidate Condorcet election, if there is always a Condorcet winner, then $f(x) = \pm x_i$ for some $i$ (dictatorship).

The case $n = 3$ follows from the below result: connects it to stability.
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**Theorem**

In a 3-candidate Condorcet election, the probability of a Condorcet winner is exactly \( \frac{3}{4}(1 - \text{Stab}_{-1/3}[f]) \).
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**Theorem**

*In a 3-candidate Condorcet election, the probability of a Condorcet winner is exactly $\frac{3}{4}(1 - \text{Stab}_{-1/3}[f])$.*

Dictator: only function for which $\text{Stab}_{-1/3}[f] = -1/3 \Rightarrow \frac{3}{4}(1 - \text{Stab}_{-1/3}[f]) = 1$. 
Peres’s Theorem

- Noise sensitivity of $f$ at $\delta$ is *probability* that misrecorded votes *change* outcome:

$$
\text{NS}_\delta[f] = \frac{1}{2} - \frac{1}{2} \text{Stab}_{1-2\delta}[f].
$$

**Theorem (Peres, 1999)**

*For any LTF $f$, $\text{NS}_\delta[f] \leq O(\sqrt{\delta})$.***
Peres’s Theorem

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**Theorem (Peres, 1999)**

*For any LTF $f$, $\text{NS}_\delta[f] \leq O(\sqrt{\delta}).$*

- $\lim_{n \to \infty} \text{NS}_\delta[\text{Maj}_n] = \frac{2}{\pi} \sqrt{\delta} + O(\delta^{3/2}).$
Applications of Peres’s theorem

Application: learning theory.

Corollary
An AND of 2 LTFs is learnable with error $\epsilon$ in time $n^{O(1/\epsilon^2)}$.

Open problem: extend Peres’s theorem to polynomial threshold functions: $\text{sgn}(p(x))$. 
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Open problem: extend Peres's theorem to polynomial threshold functions: $\text{sgn}(p(x))$. 
How to prove many of theorems: Fourier expansions, a representation of the function as a real, multilinear polynomial.

For example, \( \max_2(x_1, x_2) \), outputs the maximum of \( x_1 \) and \( x_2 \):

\[
\max_2(x_1, x_2) = \frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1 x_2.
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For a given $f$: always exists a Fourier expansion. In particular:

**Theorem**

*Every Boolean function can be uniquely expressed as a multilinear polynomial, called its *Fourier expansion*,

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S)x^S,$$

where $x^S = \prod_{i \in S} x_i$. 

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Coefficients $\hat{f}(S)$: Fourier spectrum of $f$. 

Parseval’s and Plancherel’s Theorems

Theorem (Plancherel)

For any Boolean functions \( f \) and \( g \),

\[
E[f(x)g(x)] = \sum_{S \subseteq [n]} \hat{f}(S)\hat{g}(S).
\]

Applies equally well to real-valued functions. Also yields corollary:
Parseval’s and Plancherel’s Theorems

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**Theorem (Parseval)**

For any Boolean function $f$,

$$\sum_{S \subseteq [n]} \hat{f}(S)^2 = E[f(x)^2] = 1.$$
Fourier Expansions for Stability, Influence

**Theorem**

For any Boolean function $f$ and $i \in [n]$,

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Theorem

For any Boolean function \( f \),

\[
\text{Stab}_\rho[f] = \sum_{S \subseteq [n]} \rho^{|S|} \hat{f}(S)^2.
\]
In summary:

Looked at Boolean functions in the context of social choice theory and voting. Fourier expansions of these functions along with noise stability and influence: allowed us to prove Arrow's Theorem and Peres's Theorem. Not just limited to voting theory: Learning theory. Circuit design.
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Summary and Conclusion

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Acknowledgements

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- Yufei Zhao
- MIT-PRIMES
- Our parents