

# A Probability-Based Model of Traffic Flow

Richard Yi, Harker School

Mentored by Gabriele La Nave, University of Illinois, Urbana-Champaign

January 23, 2016

## **Abstract**

Describing the behavior of traffic via mathematical modeling and computer simulation has been a challenge confronted by mathematicians in various ways throughout the last century. In this project, we introduce various existing traffic flow models and present a new, probability-based model that is a hybrid of the microscopic and macroscopic views, drawing upon current ideas in traffic flow theory. We examine the correlations found in the data of our computer simulation. We hope that our results could help civil engineers implement efficient road systems that fit their needs, as well as contribute toward the design of safely operating unmanned vehicles.

# 1 Introduction

Traffic flow is the study of how vehicles interact with each other on the road. By understanding traffic flow, we could not only increase the efficiency of road systems, allowing cars to drive closer to their desired speeds, but also make safely-operating unmanned vehicles by predicting human drivers' reactions. Furthermore, there is an intrinsic beauty in studying how tools in mathematics could be related to real-world scenarios.

Mathematicians have observed traffic flow equilibrium since the 1930s by applying a variety of methods and models. Pioneers such as Greenshields used a velocity function linearly related to density, and mathematicians have subsequently refined his model [4]. It was the team of Lighthill, Whitham, and Richards who brought to light their landmark LWR model in the 1950s, using a differential equation to describe the flow [1]. With the development of technology came increased computing power, allowing mathematicians to statistically verify their predictions with computer simulations.

This project draws upon the ideas of various existing models to form a new, simplified model. We were primarily concerned with modeling vehicles' behavior while changing lanes. This objective is often challenging because vehicles and their drivers have many distinct attributes which are not well understood. We use a computer simulation and analyze its results.

## 2 Current Models

There have been numerous methods proposed for analyzing traffic behavior. The models can generally be categorized into either microscopic or macroscopic views, each one having its own advantages and disadvantages. We will survey these models below.

### 2.1 Microscopic

Microscopic models examine the behavior of individual cars as they relate to their surroundings. Unique qualities of a car, such as its length, the driver's preferred speed, and its capacity for acceleration, are all taken into consideration. The *cellular automata model*

gives motivation for dividing up the road into discrete intervals: the road is considered as a one-dimensional array of equally-spaced intervals, and vehicles are treated as particles traveling across lattices. Velocities and distances are incremented or decremented per unit in time.

In the *car-following model*, vehicles adjust their speeds according to the cars immediately behind and in front of them, and consider surrounding cars' locations and speeds before switching lanes. **Gipps' model** gives a numerical representation of the velocity of a car which decides the feasibility of lane-changing, given its attributes and those of neighboring cars [5]:

$$v_n(t+r) = \min \left( v_n(t) + 2.5a_n r (1 - v_n(t)/V_n) \sqrt{(0.025 + v_n(t)/V_n)}, \right. \\ \left. b_n r + \sqrt{(b_n r)^2 - b_n (2(x_{n-1}(t) - s_{n-1} - x_n(t)) - v_n(t)r - v_{n-1}(t)^2/b_{n-1})} \right) \quad (1)$$

In (1), the attributes of vehicle  $n$  are the maximal acceleration  $a_n$ , maximal braking  $b_n$ , length  $s_n$ , expected velocity  $V_n$ , location  $x_n(t)$  at time  $t$ , velocity  $v_n(t)$ , and driver reaction time  $r$ .

While analyzing every vehicle gives a far more accurate representation of the traffic flow, it is computationally infeasible to track all such attributes of individual vehicles.

## 2.2 Macroscopic

Instead of observing individual cars, the macroscopic model aims to describe the traffic flow as a whole. We note characteristics of intervals, such as density and average velocity. There is no specific size for these intervals, as long as they account relatively well for deviation: if they are too small, it would resemble the microscopic model, and if they are too large, it would not be accurate enough. Macroscopic models can also be particularly useful in predicting flow behavior and even anticipating shock waves [5].

The **Lighthill-Whitham-Richards (LWR) model** is the fundamental macroscopic model, relating density and velocity at any distance  $x$  and time  $t$  [2]. It could be derived through noting the conservation of vehicles: the number of cars in a given interval  $\Delta x$

at time  $t$  is the integral of car density throughout the interval, and also equal to the difference of the fluxes at the endpoints of the interval, where flux  $J(x, t) = \rho(x, t)v(x, t)$ . Thus at any point  $x_0$  at time  $t_0$  we have [3]:

$$2\Delta x(\rho(x_0, t_0 + \Delta t) - \rho(x_0, t_0 - \Delta t)) = 2\Delta t(J(x_0 - \Delta x, t_0) - J(x_0 + \Delta x, t_0)). \quad (2)$$

Using a first-order Taylor expansion and manipulating the terms in this equality simplifies it to the form

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial J(x, t)}{\partial x} = 0. \quad (3)$$

The Greenshields model of velocity,  $v(x, t) = v_m(1 - \frac{\rho(x, t)}{\rho_m})$ , serves a fairly accurate model based on data samples from roads across the world [3]. For distance  $x$  and time  $t$ , we could use the LWR equation to express the Greenshields model in a form similar to the viscous Burgers' equation. Taking derivatives with respect to  $x$  and  $t$ , we get  $\frac{\partial v}{\partial t} = -\frac{v_m}{\rho_m} \frac{\partial \rho}{\partial t}$ ,  $\frac{\partial v}{\partial x} = -\frac{v_m}{\rho_m} \frac{\partial \rho}{\partial x}$ . Since  $\frac{\partial J}{\partial x} = \frac{\partial(\rho v)}{\partial x} = v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x}$ , we have  $\frac{\partial \rho}{\partial x} = -\frac{1}{v}(\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x})$ . After simplifying, we arrive at  $\frac{1}{v} \frac{\partial v}{\partial t} + (1 - \frac{v_m \rho}{\rho_m v}) \frac{\partial v}{\partial x} = 0$ . Thus, it appears feasible to represent car behavior as a fluid in the macroscopic model.

### 3 Probability-Based Model

The aim of this model is to provide a simple yet accurate representation of lane-changing. We focus on individual cars, but do not discretize the road and vehicles as in the cellular automata model, and do not need the numerous variables used in Gipps' model. Instead, we observe the probability a specific car will switch at a given time, which is accurate from the macroscopic point of view. This approach is reasonable because there is no definite rule to account for drivers' decisions to switch lanes: even if switching lanes is objectively correct in a given situation, the driver may not make the best decisions.

### 3.1 Background

We define a *car* (or *vehicle*) to have the following attributes at any time  $t$  on the road:

- Position  $x$ , lane  $i$ , numbered consecutively from the rightmost lane
- Preferred velocity  $V$ , and actual velocity  $v_i(x, t) = \frac{\Delta x}{\Delta t}$  on lane  $i$
- Density:  $\rho_i(x, t) = \frac{\text{Number of cars}}{\Delta x}$  on lane  $i$

We will not take into consideration any specific drivers' preferences besides their preferred velocity, and assume the length of the cars are all negligible. These factors are either irrelevant to the car's attributes we are interested in, or insignificant enough to be eliminated to minimize the computational power needed.

Our model follows the Keep Right Except to Pass (KREP) rule, widely used in traffic systems around the world. Cars stay in the rightmost lane unless the car in front of them is driving below their preferred speed. In that case, they could switch to the next lane on the left until they are able to drive at their preferred speed, and continue driving on that lane until they overtake the slower car and have enough empty distance to return to the right lane [1]. Our model will attempt to quantify the instinctive decisions that drivers make in these situations.

### 3.2 New model

The idea in this proposed model is to express lane-changing in terms of a probability. If a car in any lane (except the leftmost) is traveling at a lower speed than its desired speed because it is impeded by the car in front, then there is a chance the driver will decide to switch to the left lane. First, however, the driver must take into account safety concerns — it must be separated enough from neighboring cars to be able to switch lanes safely. The “two-second rule”, a common practice among drivers for car-following, forbids changing lanes if a car is less than two seconds behind the car in front of it or behind it. That is, for adjacent cars on locations  $x_n$  and  $x_{n+1}$ , we have  $x_{n+1} - x_n > v_n \cdot (2s)$ .

Let  $d_i(x, t)$  be the distance on lane  $i$  from the car at position  $x$  and time  $t$  from the car in front of it. We model the probability that a car at position  $x$  and time  $t$  would switch from lane  $i$  to lane  $i + 1$  as  $p_i(x, t) = \alpha \frac{d_{i+1}(x, t) - d_i(x, t)}{d_i(x, t)}$ . In this project,  $\alpha$  is taken to be a constant which factors in drivers' tendency not to switch lanes — if  $\frac{d_{i+1}(x, t) - d_i(x, t)}{d_i(x, t)}$  is small enough, the driver may prefer to stay in his lane. The probability of switching from lane  $i + 1$  back to lane  $i$  is taken to be  $kp_i$  for constant  $k$ . Since this system follows the KREP rule, it is clear that  $k > 1$ . The probabilities are also bounded between 0 and 1 (values of  $p_i$  below 0 are taken to be 0; values above 1 are taken to be 1). This is justifiable because if  $d_{i+1}(x, t) < d_i(x, t)$ , the driver would know not to change lanes, and if  $d_{i+1}(x, t)/d_i(x, t) \gg 1$ , no driver would resist switching lanes in the interest of efficiency.

## 4 Computer Simulation

Our program used the freeglut library in C++ to draw the graphics, an example of which is shown in Figure 1. Cars are represented by small rectangles, and the different colors represent what lane the cars originated from. All values we studied were displayed on the simulation as well.

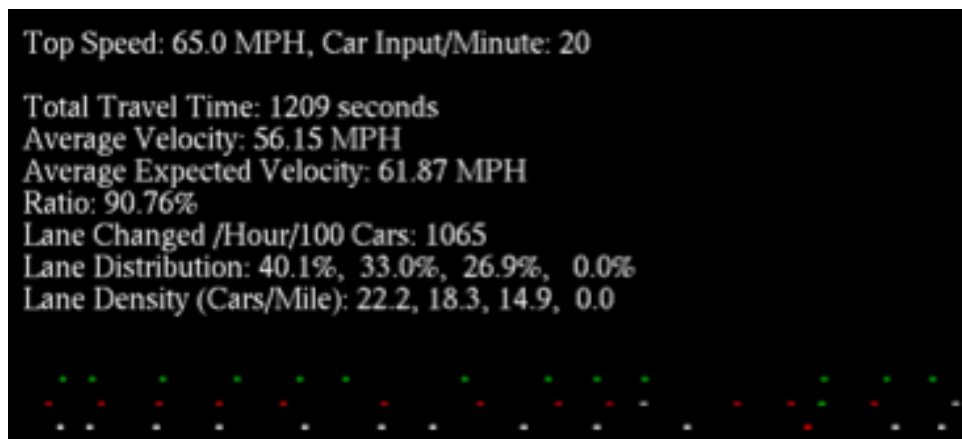
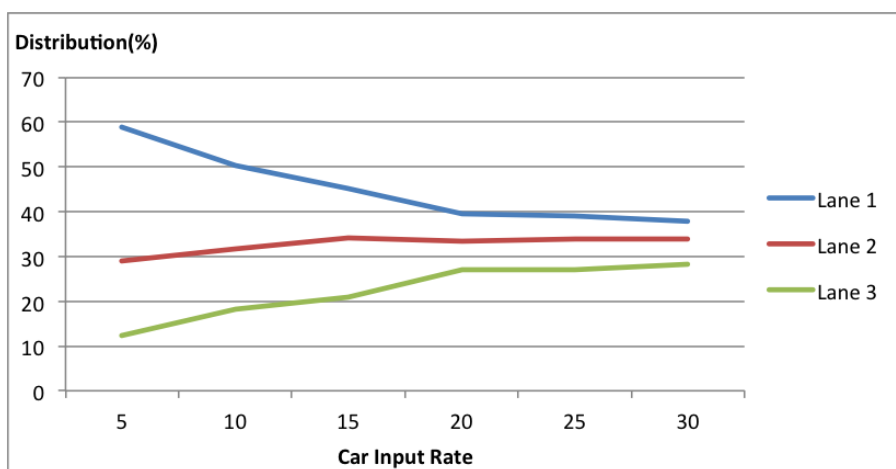


Figure 1: Screenshot of the freeglut simulation on three lanes.

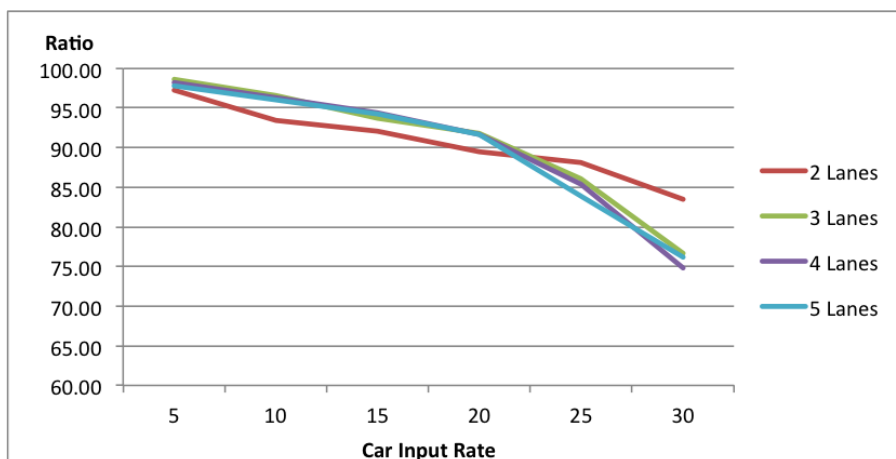
In order to run our simulation, we used reasonable values for unknown variables. The constant  $\alpha$  was taken to be 0.2 and  $k$  to be 2. Since drivers tend to prefer driving as

quickly as possible, the preferred velocity  $V$  of every car was set at 65 mph half the time, and a random value between 55 and 65 mph for the other half.

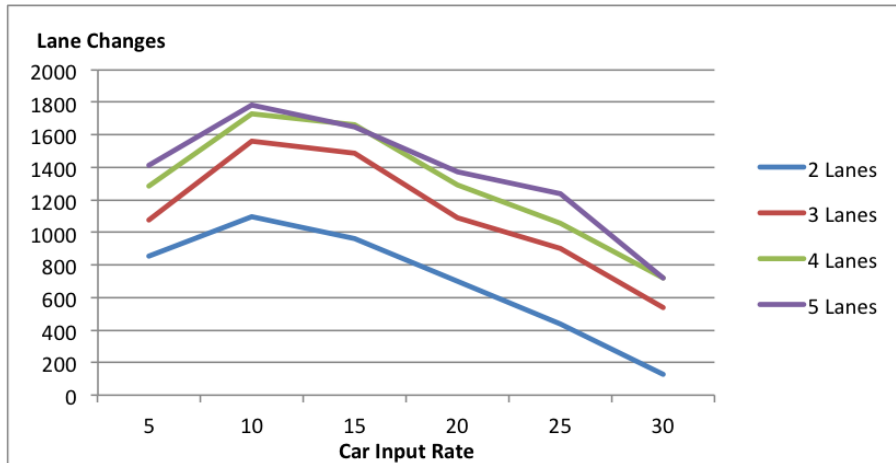
Each trial was run for twenty minutes (fast-forwarded), to ensure that the road system was stable. In Table 1, the car input rate was the number of cars per minute that entered the road on each lane, the lane distribution was a percentage, and the density was expressed in cars per mile. The ratio expressed how close the average speeds of all the cars came to their average expected speed.



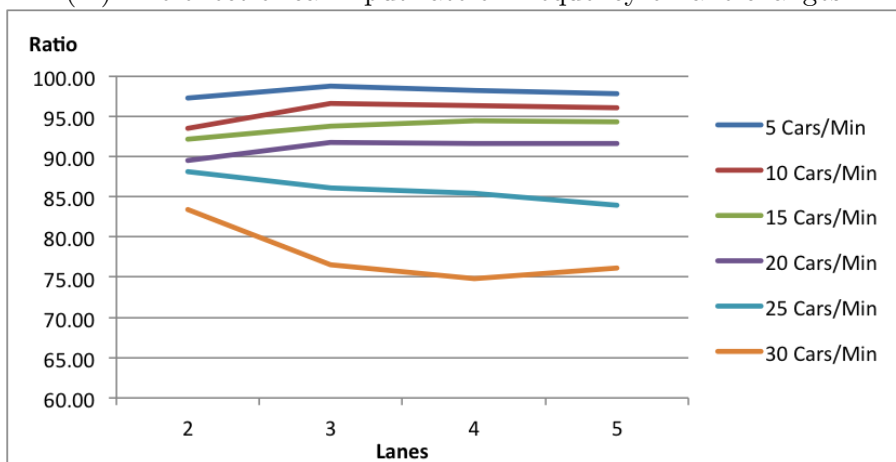
(i) Percentage distribution versus input rate on a three-lane highway.



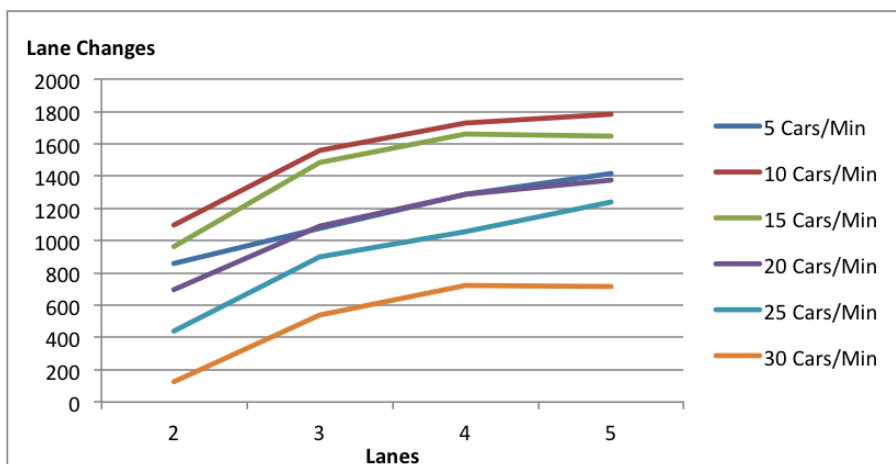
(ii) Velocity ratio versus input rate for 2-5 lanes.



(iii) The effect of car input rate on frequency of lane changes.



(iv) Velocity ratio versus number of lanes for different input rates.



(v) The effect of the number of lanes on frequency of lane changes.

Figure 2: Graphs depicting the effect of various lane settings on car distribution, velocity ratio, and frequency of lane changes.



Car Input Rate	Avg Speed	Avg Expected Speed	Ratio(%)	Car Distribution		Car Density		Lane changes/hour/100cars
				Lane 1	Lane 2	Lane 1	Lane 2	
5	60.03	61.75	97.21	68.32	31.68	6.29	2.94	855
10	57.94	61.97	93.49	62.47	37.53	12.38	7.47	1096
15	57.15	62.05	92.10	59.83	40.17	16.52	11.12	963
20	55.06	61.52	89.50	55.57	44.43	23.55	18.92	696
25	54.43	61.75	88.13	52.97	47.03	24.4	21.62	440
30	51.47	61.7	83.42	51.07	48.93	28.09	26.95	125

(i) 2 Lanes

Car Input Rate	Avg Speed	Avg Expected Speed	Ratio(%)	Car Distribution			Car Density			Lane changes/hour/100cars
				Lane 1	Lane 2	Lane 3	Lane 1	Lane 2	Lane 3	
5	61.03	61.85	98.67	58.84	28.91	12.24	8.16	4.03	1.71	1074
10	59.55	61.69	96.53	50.25	31.58	18.17	13.89	8.76	5.04	1561
15	57.74	61.62	93.70	45.09	34.04	20.87	18.7	14.16	8.68	1484
20	56.8	61.87	91.79	39.54	33.5	26.96	23.13	19.64	15.8	1091
25	53.19	61.82	86.05	38.95	33.98	27.07	26.86	23.48	18.7	899
30	47.18	61.59	76.60	37.86	33.79	28.35	31.32	27.96	23.42	537

(ii) 3 Lanes

Car Input Rate	Avg Speed	Avg Expected Speed	Ratio (%)	Car Distribution				Car Density				Lane changes/hour/100cars
				Lane 1	Lane 2	Lane 3	Lane 4	Lane 1	Lane 2	Lane 3	Lane 4	
5	60.57	61.68	98.20	48.26	29.85	15.42	6.47	8.89	5.53	2.9	1.24	1286
10	59.2	61.5	96.26	43.48	30.31	17.76	8.45	16.07	11.16	6.57	3.12	1730
15	58.31	61.75	94.43	38.26	28.61	20.81	12.33	21.21	15.89	11.56	6.85	1663
20	56.56	61.69	91.67	33.57	27.68	22.61	16.15	24.81	20.41	16.71	11.98	1291
25	52.97	62.03	85.40	31	27.51	22.7	18.79	28.59	25.32	20.97	17.38	1053
30	46.12	61.64	74.82	30.09	26.68	23.56	19.68	33.18	29.46	26.01	21.73	720

(iii) 4 Lanes

Car Input Rate	Avg Speed	Avg Expected Speed	Ratio (%)	Car Distribution					Car Density					Lane changes/hour/100cars
				Lane 1	Lane 2	Lane 3	Lane 4	Lane 5	Lane 1	Lane 2	Lane 3	Lane 4	Lane 5	
5	60.54	61.91	97.78	47.62	28.97	14.09	7.14	2.18	11.06	6.68	3.27	1.66	0.55	1415
10	59.24	61.71	96.01	39.5	28.49	18.66	9.24	4.12	18.26	13.2	8.64	4.28	1.91	1784
15	58.23	61.76	94.28	30.98	26.13	20.13	14.34	8.42	21.45	18.13	13.97	9.96	5.84	1650
20	56.62	61.76	91.67	28.64	24.42	19.65	15.88	11.41	26.46	22.56	18.11	14.67	10.54	1375
25	51.77	61.71	83.88	27.66	23.83	20.51	16	12	31.93	27.44	23.67	18.47	13.85	1236
30	47.06	61.77	76.18	25.45	22.73	19.9	17.5	14.42	35.19	31.42	27.47	24.2	19.94	718

(iv) 5 Lanes

Table 1: Increasing the car input rate would decrease the cars' average speed, increase density, and make the lane distribution more uniform.

## 5 Discussion of Results

The data in Table 1 listed the results of the set of simulations. It is evident from the three-lane data that car input rate tips the car distribution in favor of the rightmost lane; nevertheless, the density increases linearly for each lane. The distribution seems to have a common difference between consecutive lanes — as seen in the cases with an odd number of lanes  $n$ , the middle terms of the distribution values converge rapidly to  $\frac{100}{n}$ .

According to Figure 2(ii), it is evident that increasing the input rate decreases the average velocity of the cars, allowing fewer to reach their desired speeds. With more lanes, this difference becomes more significant: 30 cars entering per lane for multiple-lane roads results in all cars being able to drive at only about 75% of their desired speeds.

Though having more cars increases the density and lowers average velocity, it also does reduce the frequency of lane changes, which is another criterion for efficiency. As the proportion  $\frac{d_{i+1}-d_i}{d_i}$  decreases, cars are less likely to switch lanes. We could see why this is true: consider a large interval  $D$  on the road, and taking the cars to be uniformly distributed, we have  $D = d_i \rho_i$  for all  $i$ . Thus,  $\frac{d_{i+1}-d_i}{d_i} = \frac{\rho_i - \rho_{i+1}}{\rho_{i+1}}$ . Since the difference in densities of adjacent lanes has been noted to be approximately constant, and  $\rho_{i+1}$  is increasing, the probability of lane-changing is decreasing. There is still a discrepancy as shown in Figure 2(iii) for a very low input rate, as unobstructed cars have little need to switch lanes. Nevertheless, such a low input rate would be considered an inefficient use of road length.

Increasing the number of lanes while keeping the input rate fixed does not have a large impact on drivers' velocities, as seen in the stable ratio in Figure 2(iv). Unless the density is very high, more lanes could have only a beneficial effect on cars' velocities.

It is remarkable from Figure 2(v) that having more lanes also increases the frequency of lane changes, regardless of input rate. Since each lane is used, adding another lane would inevitably give cars who would be on the leftmost lane an extra option to switch. This brings to question how to optimize the number of lanes on a road — more lanes do bring cars closer to their desired speeds, but they also are costlier and prompt more

lane-switching which reduces their efficiency.

The credibility of the results could be verified by noting that the correlation between car input rate and average speed, distribution, and density follow reasonable patterns. Moreover, the velocity ratio stays high for low densities and gradually decreases as the density increases, as expected.

## 6 Further Research

Since this is a working model, it could be improved further to fit in with realistic data. Foremost, the constants we used, such as setting  $\alpha$  at 0.2 and  $k$  at 2, could be modified. As we were not able to access relevant data on a genuine traffic system, we could use only approximations we deemed acceptable.

There is also the problem of safety that needs to be addressed. Ideally, drivers would be able to evaluate surrounding factors such as density and velocity of neighboring cars, as in an automated vehicle. Without this precision, however, they would be bound to make mistakes. Our model has already eliminated cases where drivers are unreasonably close: the “two-second” rule. But humans do occasionally get into dangerous situations, such as by unexpected braking — another study had approximated it to be around once per 5000 kilometers driven [1].

## 7 Acknowledgments

I would like to thank Prof. Gabriele La Nave for taking the time to answer my questions and Dr. Tanya Khovanova for offering advice and guidance throughout the research process. In addition, I am grateful to the PRIMES-USA program for this unique research opportunity.

## References

- [1] Control #31184. The keep-right-except-to-pass rule. 2014.
- [2] S. Childress. Notes on traffic flow. 2005.
- [3] M. H. Holmes. *Introduction to the Foundations of Applied Mathematics*. 2009.
- [4] S. Maerivoet and B. de Moor. Traffic flow theory. *arXiv:physics/0507126*, 2005.
- [5] M. Treiber and A. Kesting. *Traffic Flow Dynamics: Data, Models and Simulation*. 2013.