On the Extremal Functions of Multi-dimensional Matrices

Peter Tian
Mentor: Jesse Geneson
Fourth Annual MIT PRIMES Conference
May 17th, 2014
0-1 matrix A contains 0-1 matrix B

A

submatrix

Changing 1s to 0s

B
Extremal Function of 0-1 Matrices

• By deleting rows and columns and changing some 1s to 0s, $A$ contains $B$.

• Otherwise we say $A$ avoids $B$

• Let $\text{ex}(n, P)$ be the maximum number of one entries in a $n \times n$ matrix avoiding $P$
Some background

- What are all matrices $P$ such that $\text{ex}(n, P) = O(n)$? [FH]
- $\text{ex}(n, P) = O(n)$ for permutation matrices $P$ [MT]
- $\text{ex}(n, P) = O(n)$ for double permutation matrices $P$ [G]
$d$-dimensional

- $d$-dimensional 0-1 matrix
  \[ M = (M; n_1, n_2, \ldots, n_d) \text{ where } M \subseteq [n_1] \times [n_2] \times [n_3] \times \cdots \times [n_d] \]

- extremal function $f(n, P, d)$ is the maximum number of ones in an $n \times \cdots \times n$ $d$-dimensional 0-1 matrix that avoids the $d$-dimensional matrix $P$

- $f(n, P, 2) = \text{ex}(n, P)$
More Questions

• What are all matrices $P$ such that $f(n, P, d) = O(n^{d-1})$?

• $f(n, P, d) = O(n^{d-1})$ for all d-dimensional permutation matrices $P$ [KM]

Permutation matrix with $d = 3, k = 4$
Theorem 1

• Tuple Permutation matrix

• \( f(n, j, k, d) = \max_P f(n, P, d) \), where \( P \) ranges over \( j \)-tuple permutation matrices of size \( k \)

Theorem 1: \( f(n, j, k, d) = O(n^{d-1}). \)
Proof of Theorem 1

• Let $A$ be an $sn \times \cdots \times sn$ matrix that avoids $2k \times k \times \cdots \times k$ double permutation matrix $P$

• An $i$-row is a maximum set of entries $(x_1, x_2, \ldots, x_d)$ with only $x_i$ varying
Proof of Theorem 1

- Divide $A$ into $n^d$ blocks of size $s \times \cdots \times s$

$n = 4$, $s = 2$, and $d = 2$
Proof of Theorem 1

• **Wide Chunks** have at least $2k$ one entries in the same 1-row

• A wide chunk has one non-empty block

• **$j$-tall chunks** have at least $k$ one entries with distinct coordinates in the $j$th dimension
Proof of Theorem 1

The maximum number of 1s in $A$ is:

$$f(sn, 2, k, d) \leq s^d \left[ n \binom{s}{2k} f(n, 1, k, d - 1) \right]$$

(max number of 1s from wide chunks in $A$)

$$+ (2k - 1)s^{d-1} \left[ (d - 1)n \binom{s}{k} f(n, 1 + s^{d-2}, k, d - 1) \right]$$

(max number of 1s from j-tall but non wide chunks in $A$)

$$+ (2k - 1)(k - 1)^d \left[ f(n, 2, k, d) \right]$$

(max number of 1s from non-wide, non-tall chunks in $A$)
Proof of Theorem 1

- By induction on $n$ and $d$, $f(n, 2, k, d) = O(n^{d-1})$

- By induction on $j$, $f(n, j, k, d) = O(n^{d-1})$
Block Permutation Matrices

- Block matrices $R^{k_1,k_2,...,k_d}$

- Block permutation matrices $P^{k_1,k_2,...,k_d}$
Theorem 2

Theorem 2: \( f(n, R^{k_1, k_2, \ldots, k_d}, d) = O(n^{d - \frac{\max(k_1, k_2, \ldots, k_d)}{k_1 k_2 \ldots k_d}}) \)

- **Base Case:** Kovari, Sos, and Turan proved this for \( d = 2 \)
Lower bound on blocks

Theorem 3: \( f(n, R^{k_1, \ldots, k_d}, d) = \Omega(n^{d - \frac{k_1 + k_2 + \cdots + k_d - d}{k_1 k_2 \cdots k_d - 1}}) \).

- A function is unboundedly super \( n^{d-1} \) if for all \( k \) there exists \( c \) such that for all \( n \), \( f(cn) > k c^{d-1} f(n) \).

- \( n^{d-1+\epsilon} \) for \( \epsilon > 0 \) is unboundedly super \( n^{d-1} \).
Tensor product

- $P \otimes Q$ is the matrix obtained by replacing each 1 of $P$ with a copy of $Q$

**Lemma:** If $P$ is a $d$-dimensional permutation matrix and $Q$ is a matrix such that $f(n, Q, d)$ is unboundedly super $n^{d-1}$ then $f(n, P \otimes Q, d) = \Theta(f(n, Q, d))$
Block permutation matrix

- $\mathbf{P}^{k_1,k_2,\ldots,k_d} = \mathbf{P} \bigotimes \mathbf{R}^{k_1,k_2,\ldots,k_d}$

**Corollary:** If $\mathbf{P}$ is a permutation matrix, then

$f(n, \mathbf{P}^{k_1,k_2,\ldots,k_d}, d) = \Theta(f(n, \mathbf{R}^{k_1,k_2,\ldots,k_d}, d))$
Open Problems

• What are all $d$-dimensional matrices $P$ such that $f(n, P, d) = O(n^{d-1})$?

• What are all $d$-dimensional matrices $P$ such that $f(n, P, d)$ is unboundedly super $n^{d-1}$?

• What are tight bounds on $f(n, R^{k_1, k_2, \ldots, k_d}, d)$ and $f(n, P^{k_1, k_2, \ldots, k_d}, d)$?
References

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Acknowledgements

• My mentor Jesse Geneson

• MIT PRIMES USA

• My family