Linear Extensions of Directed Acyclic Graphs

Kyle Gettig

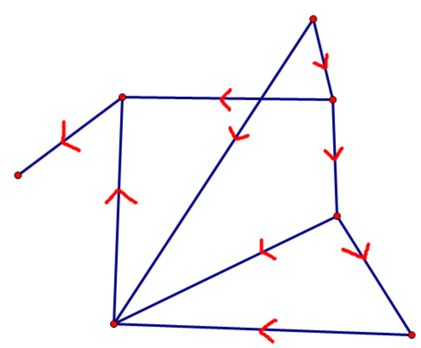
Mentor Benjamin Iriarte

Fourth Annual MIT PRIMES Conference

May 17, 2014

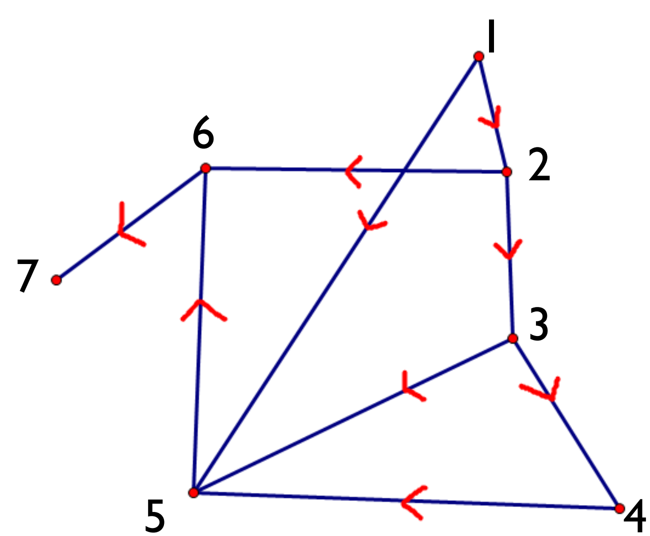
Motivation

- Individuals can be modeled as vertices of a graph, with edges connecting individuals who have met at some point in time.
- If each member contracts a disease at a different time, the edges of the graph can be oriented towards the individual that contracted the disease later in time. This produces an acyclic orientation.



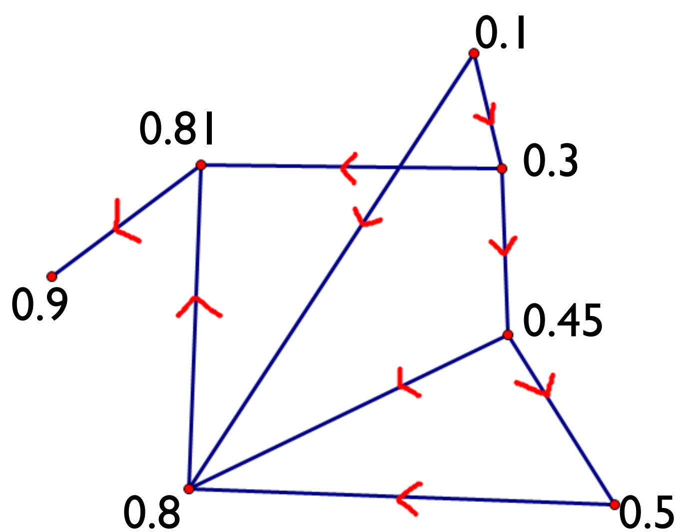
Motivation

- The orientation with the maximum number of linear extensions gives the most probable partial ordering using the edges of the graph.



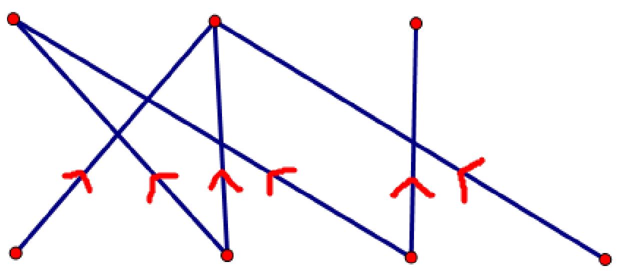
Motivation

- Similar situations can also be modeled similarly. If each vertex is assigned a random number between 0 and 1, and each edge is directed towards its largest vertex, the orientation with the largest number of linear extensions gives the most probable orientation.



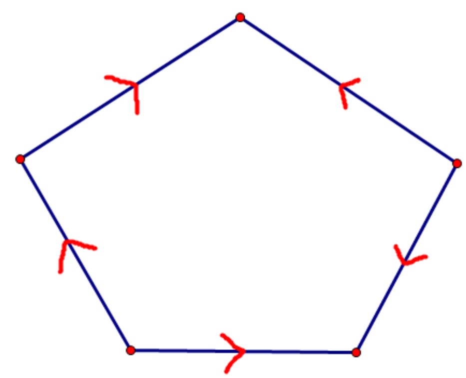
Known Results

- Given a bipartite graph, the optimal orientations are exactly the bipartite orientations, that is, the orientations with no directed two-paths.



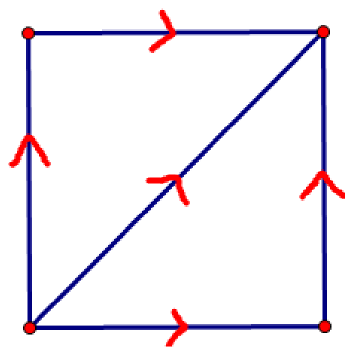
Known Results

- Given an odd cycle, the optimal orientations are exactly those with only one directed two-path.

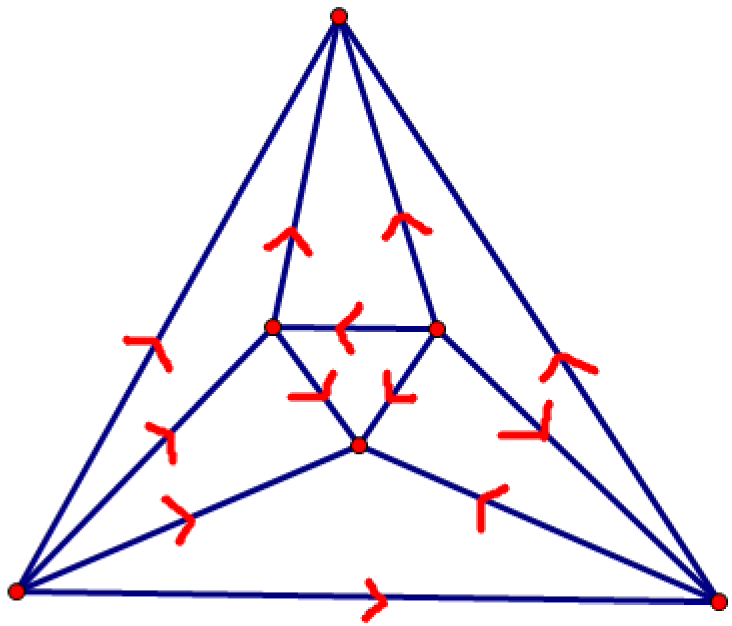


Known Results

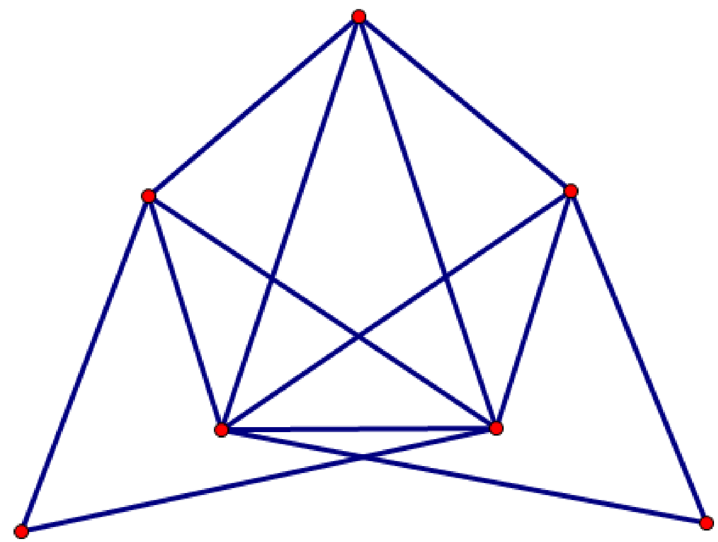
- *Transitive orientations* of a graph are orientations in which two vertices are comparable iff there is an edge connecting them.
- A *comparability graph* is a graph for which a transitive orientation exists.
- Given a comparability graph, the optimal orientations are exactly the transitive orientations.



Comparability Graphs



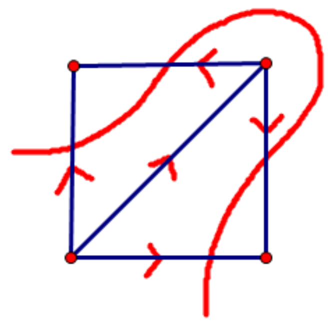
More complex comparability graph, with transitive orientation



Non-comparability graph; no such transitive orientation exists

Max-Cut Problem

- Given a graph, the *Max-Cut Problem* asks to find a partition of the vertices of the graph into two subgraphs such that the number of edges connecting two vertices not in the same subgraph is maximized.



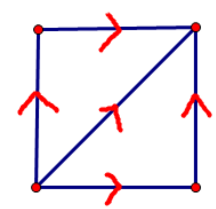
- Given a solution to the max-cut problem, an orientation is called *bipartite* if, when the graph is restricted to edges only between the two subgraphs, the orientation is bipartite.

Max-Cut Problem

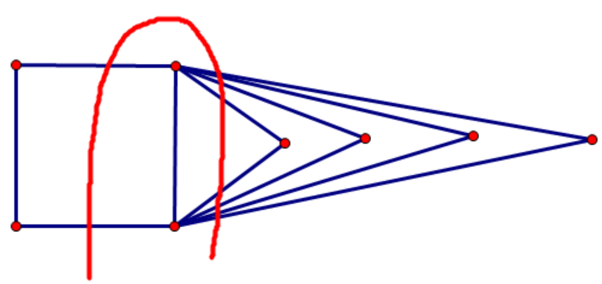
- **Conjecture:** Given a graph and a solution to the Max-Cut Problem, there exists an orientation of the edges of the graph that is both bipartite with respect to this solution and also maximizes the number of linear extensions.

Max-Cut Problem

- The converse is not necessarily true; the oriented graph does not have a solution to the Max-Cut Problem such that the orientation is bipartite with respect to it.



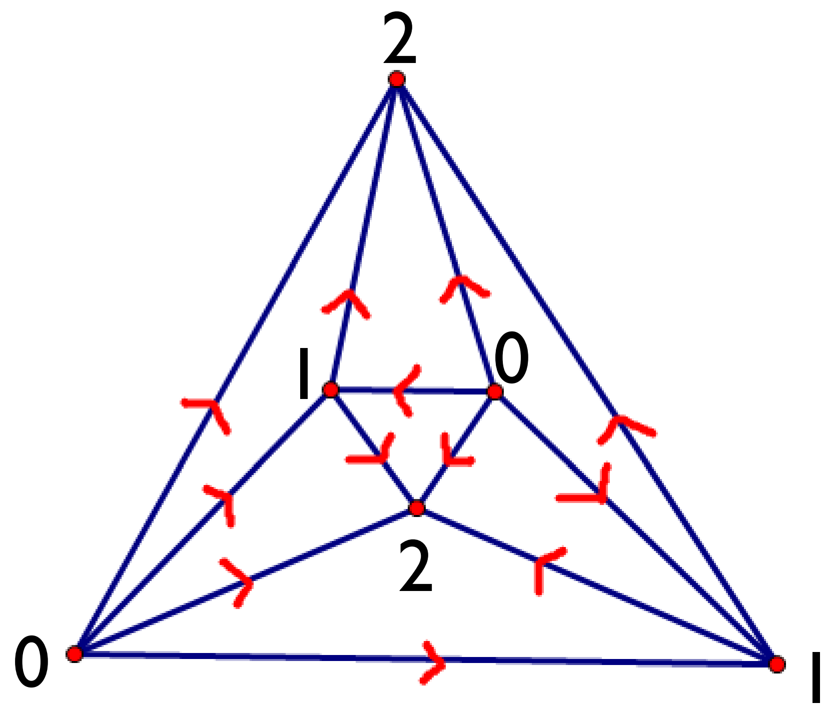
- Unfortunately, the conjecture itself is not true either. The below graph and solution to the Max-Cut Problem does not have a transitive orientation bipartite with respect to that solution.



Induced Colorings

- For each acyclic orientation of a graph, each vertex can be assigned a number equal to the length of the maximum directed path ending at that vertex.
- Partitioning the vertices based on this number creates a proper coloring of the vertices (Gallai-Hasse-Roy-Vitaver).
- **Conjecture:** Given a graph, does an acyclic orientation that maximizes the number of linear extensions also induce a minimal proper coloring?

Induced Colorings (Example)



Each vertex is assigned a number equal to the maximum length of all directed paths that end on that vertex. This induces a proper coloring with three colors, which is minimal since the graph contains a three-cycle.

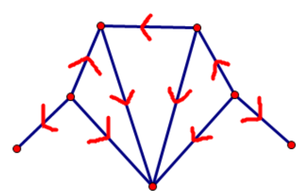
Induced Edges

- Given a graph and an acyclic orientation, if two vertices are comparable and there is not an edge between them, it can be said that they create an induced edge.

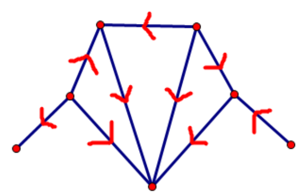
- **Conjecture:** Given a graph and an acyclic orientation that maximizes the number of edges, the orientation also minimizes the number of induced edges.

Small Non-Comparability Graphs

- Using more specific enumeration techniques, the number of linear extensions were calculated for some small non-comparability graphs. Being the simplest graphs for which the optimal orientations are not known, they may provide more insight into the previous two conjectures.

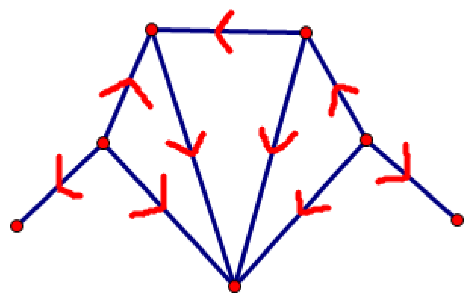


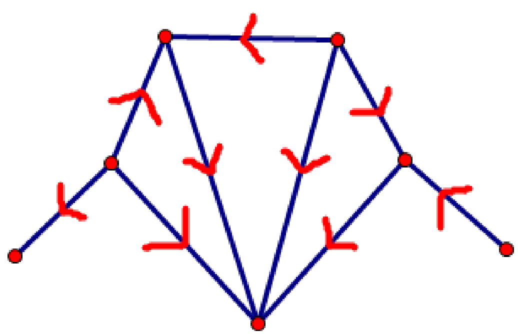
66 Linear Extensions



77 Linear Extensions

Small Non-Comparability Graphs





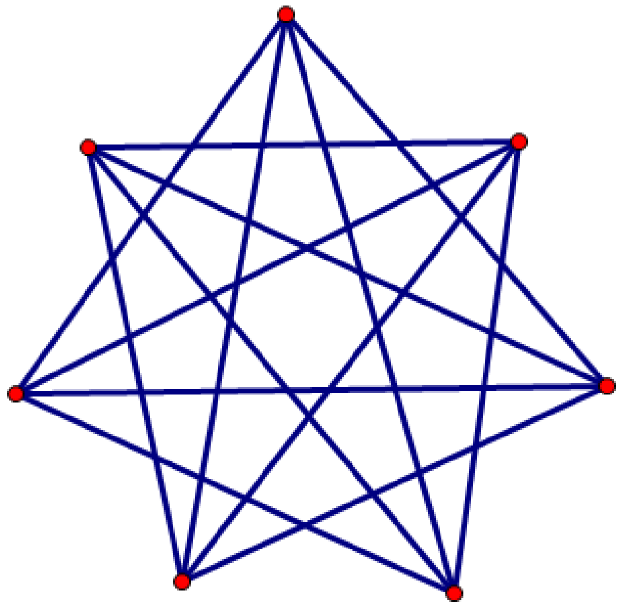
- Casework on the values of the bottom three vertices
- The maximal element must be either the bottom or bottom-left vertex

  This determines an alternating path; the number of linear extensions is then the associated Euler Number

Small Non-Comparability Graphs

- An *odd anti-cycle* is a graph whose complement is an odd cycle.
- A *perfect graph* is a graph whose chromatic number is equal to its maximum clique size.
- An *induced subgraph* of a graph is a subgraph such that any edge in the graph connecting two vertices of the subgraph is also in the subgraph.
- A graph is perfect iff it does not have a subgraph that's either an odd cycle or an odd anti-cycle with at least 5 vertices (Strong Perfect Graph Theorem; Chudnovsky, Robertson, Seymour, Thomas)

Small Non-Comparability Graphs



- An odd anti-cycle of 7 vertices
- Further investigations will be into optimal orientations of these graphs

Generalizations to Random Graphs

- Since in many modeling problems, it is not known with absolute certainty whether or not an edge exists, a random graph may be used, in which each edge is associated with a probability.
- A random graph can be thought of as a set of $2^{\binom{n}{2}}$ fixed subgraphs, each also associated with a probability:

$$\prod_{e \in E} p_e \prod_{f \notin E} (1 - p_f),$$

where E is the set of edges existing in the subgraph, and p_e is the probability of edge e .

Generalizations to Random Graphs

- An acyclic orientation of a random graph can be defined as an acyclic orientation of a complete graph on n vertices restricted to the edges of each subgraph.

Generalizations to Random Graphs

- The number of linear extensions of a random graph can be defined to be the expected number of linear extensions for each of its subgraphs:

$$\sum_{G \in S} p_G \epsilon_O(G),$$

where p_G is the probability of subgraph G, $\epsilon_o(G)$ is the number of linear extensions of G, and the summation is taken over all $2^{\binom{n}{2}}$ subgraphs.

By linearity of expectation, this is equal to

$$\sum_{o \in O} p_o \cdot n(o$$

where O is the set of all $n!$ total orderings on n vertices, $n(o)$ is the number of subgraphs of G for which o is a possible linear extension, and p_o is the sum of the probabilities of these subgraphs.

Generalizations to Random Graphs

- We can define a random bipartite graph to be a random graph for which there exists a partition of the vertices into two classes such that the probability of an edge between two vertices in the same class is 0 (or effectively 0, for modeling purposes). In this case, the orientations of the edges between two such vertices can be ignored.

- For a random bipartite graph, the orientations that maximize the number of linear extensions are exactly the bipartite orientations.

Generalizations to Random Graphs

- In general, however, finding optimal orientations for random graphs is much more difficult than finding them for fixed graphs, so further investigations should focus more on smaller cases for fixed graphs.

Thanks

- MIT PRIMES
- My mentor, Benjamin Iriarte
- My parents