A Diagrammatic Approach to the $K(\pi, 1)$ Conjecture

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WHAT IS A COXETER GROUP?
**What is a Coxeter group?**

**Definition**
A Coxeter group is given by generators $g_1, g_2, \ldots, g_n$ with relations:

- $g_i^2 = 1$ for all $i$
- $(g_ig_j)^{m_{ij}} = 1$ for all $i, j$
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**Definition**

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Some examples of Coxeter groups include the symmetric group and reflection groups.
**COXETER DIAGRAMS**

Coxeter diagrams can be used to visualize Coxeter groups.

- Each vertex represents a generator
- Edges show the relations between generators
We can use Coxeter groups to create certain graphs.
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- Each color represents a generator.
- The degree of each vertex is determined by the relations between generators.
**$K(\pi, 1)$ Conjecture**

There is a conjecture known as the $K(\pi, 1)$ conjecture regarding the second homotopy group of the dual Coxeter complex.

- The dual Coxeter complex is a topological space associated to each Coxeter group
- Elements of second homotopy group correspond to aforementioned graphs

Proving this conjecture is equivalent to proving that all possible graphs for a Coxeter group can be simplified to the empty graph using a sequence of allowed moves.
MOVES ON DIAGRAMS

How can we simplify a graph?
HOW CAN WE SIMPLIFY A GRAPH?

3 ALLOWABLE MOVES:

- Circle relation
- Bridge relation
- Zamolodchikov relations
**MOVES ON DIAGRAMS**

How can we simplify a graph?

3 allowable moves:

- Circle relation
- Bridge relation
- Zamolodchikov relations
**Circle Relation**

We are allowed to add or remove empty circles of any color.
**Bridge Relation**

If we have two edges of the same color, we can switch around which vertices they connect to, as long as we do not create any new intersections.
ZAMOLODCHIKOV Relations (ZAM Relations)

Zam relations vary for different coxeter groups. They are found through the reduced expression graph for the longest element.
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Our Project

In our project, our primary goal was to prove the $K(\pi, 1)$ conjecture for specific Coxeter groups.
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- $I_2(m)$
- $A_3$
- $B_3$
- $G \times H$
- Directed cases
- Working on $A_n$
**Adjacent Vertices**

**Theorem**

*We can remove adjacent vertices of the same type.*
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**Proof.**

![Diagram](image-url)
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Proof.

- Only 1 type of vertex so necessarily 2 adjacent vertices of same type
Theorem

*The family of Coxeter groups* $I_2(m)$ *satisfies the* $K(\pi, 1)$ *conjecture.*

**Proof.**

- Only 1 type of vertex so necessarily 2 adjacent vertices of same type
- Use induction on number of vertices
\( A_3 \)

\textbf{Theorem}

*The Coxeter group \( A_3 \) satisfies the \( K(\pi, 1) \) conjecture.*
Theorem

*The Coxeter group $A_3$ satisfies the $K(\pi, 1)$ conjecture.*

Strategy: Look at subgraph of blue color and use Euler characteristic: $V + F = E + 2$ to find a small face.

- Delete the small face.
Theorem

The Coxeter group $A_3$ satisfies the $K(\pi, 1)$ conjecture.

Strategy: Look at subgraph of blue color and use Euler characteristic: $V + F = E + 2$ to find a small face.

- Delete the small face.

Using parity, the only nontrivial case is a blue face with 4 edges.
Any face with 4 edges can be transformed into ZAM for $A_3$.

- Look at continuation of edges outside of face
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- Look at continuation of edges outside of face
- Use bridge relation to connect edges
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- Look at continuation of edges outside of face
- Use bridge relation to connect edges

- After using Zam transform, there must be adjacent vertices of the same type.
Theorem

The Coxeter group $B_3$ satisfies the $K(\pi, 1)$ conjecture.
$B_3$

$B_3$:  

Theorem

The Coxeter group $B_3$ satisfies the $K(\pi, 1)$ conjecture.

- We examine the subgraph of green color.
Theorem

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- Use Euler characteristic to find a small green face.
Theorem

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- We examine the subgraph of green color.
- Use Euler characteristic to find a small green face.
- Faces with odd number of edges necessarily have adjacent vertices of the same type that can be removed. Using parity and more complicated arguments, faces with 2 or 4 edges also necessarily have vertices that can be removed.
Theorem

*The Coxeter group $B_3$ satisfies the $K(\pi, 1)$ conjecture.*

- We examine the subgraph of green color.
- Use Euler characteristic to find a small green face.
- Faces with odd number of edges necessarily have adjacent vertices of the same type that can be removed. Using parity and more complicated arguments, faces with 2 or 4 edges also necessarily have vertices that can be removed.
- Only nontrivial case is a green face with 6 edges. Vertices of type green-red and green-blue alternate around face.
Using idea that no adjacent vertices can be of same type, we can manipulate this face into the $B_3$ ZAM relation.

- Examine continuation of edges outside of face.
Using idea that no adjacent vertices can be of same type, we can manipulate this face into the $B_3$ ZAM relation.

- Examine continuation of edges outside of face.
- Only 1 vertex of type red-blue inside face.
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- Examine continuation of edges outside of face.
- Only 1 vertex of type red-blue inside face.
- Use bridge relation inside and outside of face to connect edges.
- Use ZAM transformation and get adjacent vertices of the same type.
$G \times H$

**Theorem**

*If the $K(\pi, 1)$ conjecture holds for groups $G$ and $H$, then it holds for the group $G \times H$.***
Theorem

If the $K(\pi, 1)$ conjecture holds for groups $G$ and $H$, then it holds for the group $G \times H$.

Strategy: Commutative Colors
Theorem

If two generators commute, then we can move the edges corresponding to them independently.
$A_1 \times H$

**Theorem**

*If two generators commute, then we can move the edges corresponding to them independently.*

Using this idea, we can solve the general case $G \times H$ by essentially separating the graph formed by the generators of $G$ from the one formed by the generators of $H$. 
We also solved some cases involving oriented graphs.
Oriented cases are much more difficult because we cannot necessarily remove adjacent vertices of the same type.
**Oriented** $A_2$

Oriented cases are much more difficult because we cannot necessarily remove adjacent vertices of the same type.

Strategy: Look at the longest cycle
We can take this project in multiple directions in the future.

- We could continue proving the $K(\pi, 1)$ conjecture for other Coxeter groups.
- We could generalize our proofs to classes of Coxeter groups. (For example, we have a nearly-finished proof for $A_n$.)
- We could also investigate oriented versions of the cases we have already solved.
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