Geodesics in the Hypercube

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Colorings of the Cube

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- Notice how we can always find a monochromatic path between two opposite points.
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Also notice how this monochromatic path cycles.
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We similarly define the antipodal edge of $xy$ as $x^ay^a$.

An antipodal coloring of $Q_n$ is one where no antipodal edges are the same color.
A geodesic on $Q_n$ is the shortest possible path between two vertices. In other words, it is a path that traverses each coordinate direction at most once. An antipodal geodesic is one between antipodal vertices.
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Conjecture (Leader and Long, 2013)

Given an antipodal 2-coloring of $Q_n$, there exists a monochromatic geodesic between some pair of antipodal vertices.
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It has been shown that these two conjectures are equivalent.
Examples of Conjecture 2

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Outline of our Work

We took these conjectures and explored two areas:

1. We showed that they were true for the cases $n = 2, 3, 4, 5, 6$.
2. We looked at the opposite problem, maximality, in the following cases:
   - Antipodal 2-colorings of the cube
   - Subgraphs of the cube with a fixed proportion of edges
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1. Antipodal 2-colorings of $Q_n$

2. Subgraphs of $Q_n$ with a fixed number of edges
Maximal Antipodal 2-colorings: Idea

We aim to maximize the number of monochromatic geodesics.

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A **subcube 2-coloring** of $Q_n$ colors the edges of disjoint $n - 1$-dimensional subcubes in $Q_n$ opposite colors, and then colors antipodally the remaining edges connecting these subcubes.
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We conjectured that such a subcube coloring contained the maximum number of geodesics.
Theorem

The maximum number of geodesics in an antipodal 2-coloring of $Q_n$ is $2^{n-1}(n-1)!$, which occurs **only in a subcube coloring**.
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1. Antipodal 2-colorings of $Q_n$

2. Subgraphs of $Q_n$ with a fixed number of edges
Idea: without an antipodal coloring, best way to maximize is to pack monochromatic cycles.

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Subgraphs of the Cube: Idea

- Idea: without an antipodal coloring, best way to maximize is to pack monochromatic cycles.
- Cycles have the most geodesics for the number of edges
- This led us to the configuration below: a subgraph containing all edges in the 'middle layer'

![Diagram of a cube with vertices labeled 000, 001, 010, 011, 100, 101, 110, 111, with edges connecting them to form a hexagon. The cube is also shown with its vertices at the corners.](image)
Let $d(v)$ be the number of 1’s in the coordinate form of $v$.

**Definition**

A **middle-layer subgraph** is one containing an edge $E = \{v_1, v_2\} \in Q_n$ if and only if $\frac{n}{2} - C \leq d(v_1), d(v_2) \leq \frac{n}{2} + C$, where $C$ depends on the proportion of edges.
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![Diagram](Image of a cube with edges highlighting the middle layer)

**Edges concentrated in the 'middle layer'**

(1, 1, ..., 1)  
(0, 0, ..., 0)

**Shown is a path of edges in this middle layer.**
Subgraphs of the Cube: Computation

We calculate the maximal number of antipodal geodesics in a subgraph with a fixed proportion of edges.
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**Result**: Given that our proportion of edges is equivalent to the area shown below:
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**Result:** Given that our proportion of edges is equivalent to the area shown before, the proportion of geodesics in a middle layer subgraph is equivalent to the area shown below:
Future Directions

- Work on a similar problem, except for antipodal subgraphs of the hypercube
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- Explore the original conjectures further
- Look into similar results or applications to other regular graphs besides the hypercube
- Incorporate probability into these colorings: e.g. the expected number of antipodal geodesics
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