$q$-Analogues of Symmetric Polynomials and nilHecke Algebras

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Symmetric Functions

Definitions

Define the elementary symmetric functions by:

\[ e_k(x_1,\ldots,x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \]

Define the complete homogenous symmetric functions by:

\[ h_k(x_1,\ldots,x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k} \]

\[ h_2(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3 \]
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Goals and Motivation

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1. To develop a $q$-analogue of symmetric functions.
2. The ”odd” ($q = -1$) nilHecke algebra can be used in categorification of quantum groups.
   We expect that our $q$-analogue can also be used in categorification.
3. Our $q$-bialgebra also has connections to 4D-topology.
Introduction to $q$-Bialgebras

**Definition: Algebra**

An *algebra* $A$ is characterized by the following two maps:

$$\eta : \mathbb{C} \to A$$

$$m : A \otimes A \to A$$
Definition: Algebra

An algebra \( A \) is characterized by the following two maps:

\[
\eta : \mathbb{C} \rightarrow A
\]

\[
m : A \otimes A \rightarrow A
\]

\( q \)-Swap and Identity Maps

\[
\tau : v \otimes w \rightarrow q^{|v||w|} w \otimes v
\]

\[
1_A : A \rightarrow A
\]
Introduction to $q$-Bialgebras

Multiplication

We define the multiplication on $A \otimes A$ by

$$(a \otimes b)(c \otimes d) = q^{\|b\| \|c\|}(ac \otimes bd)$$
Introduction to $q$-Bialgebras

Multiplication

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$$(a \otimes b)(c \otimes d) = q^{||b||^{c}}(ac \otimes bd)$$

Multiplication map $m_2$

The multiplication map $m_2: A^{\otimes 4} \to A^{\otimes 4}$ is

$$m_2 = (m \otimes m)(1_A \otimes \tau \otimes 1_A)$$
Introduction to $q$-Bialgebras

Definition: Coalgebra

A coalgebra has the following maps:

\[ \epsilon : A \to \mathbb{C} \]
\[ \Delta : A \to A \otimes A \]
Definition: Coalgebra

A coalgebra has the following maps:

\[ \epsilon : A \rightarrow \mathbb{C} \]
\[ \Delta : A \rightarrow A \otimes A \]

Definition: Bialgebra

A bialgebra has all four maps \( \eta, m, \epsilon, \) and \( \Delta, \) with the added compatibility that the comultiplication is an algebra homomorphism.
Description as a $q$-Bialgebra

Let $N^\Lambda_q$ be a free, associative, $\mathbb{Z}$-graded $C$-algebra with generators $h_1, h_2, ...$. Let $q \in \mathbb{C}$.

We define $h_0 = 1$, $h_i = 0$ for $i < 0$, and $\deg(h_k) = k$.

We define $h_{\lambda} = h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_r}$.

Define multiplication as:

$$(w \otimes x)(y \otimes z) = q^{\deg(x) \deg(y)} (wy \otimes xz).$$

Define comultiplication as:

$$\Delta(h_n) = \sum_{m=0}^{n} h_m \otimes h_{n-m}.$$
Quantum Noncommutative Symmetric Functions

Description as a $q$-Bialgebra

Let $\Lambda^q$ be a free, associative, $\mathbb{Z}$-graded $\mathbb{C}$-algebra with generators $h_1, h_2, \ldots$. Let $q \in \mathbb{C}$. 

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- We define $h_0 = 1$, $h_i = 0$ for $i < 0$, and $\text{deg}(h_k) = k$.
- We define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_r}$.
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Diagrammatics for the Bilinear Form

Let’s consider the method to determine \((h_1 h_2 h_1, h_2 h_2)\).
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**Rules**

There are no triple intersections, no critical points with respect to the height function, no instances of two curves intersecting at two or more points, and no crossing between curves originating from the same platform.
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\[(h_1h_2h_1, h_2h_2) = 1 + 2q^2 + q^3\]
Definition
Define $\text{Sym}^q \cong N\Lambda^q / R$, where $R$ is the radical of the bilinear form.

- The ”odd case” refers to $q = -1$, studied in [EK].
- The ”even” case refers to $q = 1$, studied in [GKLLRT].
Definition

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Diagrammatic Property

1. No strands from different tensor factors intersect:

$$ (w \otimes x, y \otimes z) = (w, y)(x, z). $$
The Elementary Symmetric Functions

Definitions

Inductively define \[ \sum_{k=0}^{n} (-1)^k \binom{k}{2} h_{n-k} e_k = 0 \]
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Inductively define \( \sum_{k=0}^{n} (-1)^k q^{(k)} h_{n-k} e_k = 0 \)

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We will use a blue platform with \( k \) strands to denote \( e_k \).
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Theorem

\( (h_\lambda, e_k) = 0 \) if \( |\lambda| = k \), unless \( \lambda = 1^k \).
Diagrammatics for the Bilinear Form

Idea of Proof

\[(h_{m \times e})_n = \begin{cases} (x,e)_{n-1} & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}\]

Use strong induction on \(n\) to find \((h_{m \times e})_k (h_n - k)\).

By definition:

\[(-1)^{n+1} q(n^2) (h_{m \times e})_n = \sum_{k=0}^{n-1} (-1)^k q(k^2) (h_{m \times e})_k (h_n - k)\]
Diagrammatics for the Bilinear Form

Idea of Proof

- Show that

\[ (h_m x, e_n) = \begin{cases} 
  (x, e_{n-1}) & \text{if } m = 1 \\
  0 & \text{otherwise} 
\end{cases} \]
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\end{cases}\]

- Use strong induction on \(n\) to find \((h_m x, e_k h_{n-k})\)

- By definition:

\[(-1)^{n+1} q\binom{n}{2} (h_m x, e_n) = \sum_{k=0}^{n-1} (-1)^k q\binom{k}{2} (h_m x, e_k h_{n-k})\]
There are two cases to consider by the inductive hypothesis applied to $k < n$. Either there is a strand connecting $h_m$ and $e_k$, or there is not.
Diagrammatics for the Bilinear Form

Idea of Proof

If no strand connects $h_m$ and $e_k$. This contributes $q^{km}(x, e_k h_{n-k-m})$. 
Diagrammatics for the Bilinear Form

Idea of Proof

If a strand connects $h_m$ and $e_k$. This contributes $q^{(k-1)(m-1)}(x, e_{k-1} h_{n-k-m+1})$. 
Summary of Diagrammatic Rules for any $q$

Theorem

$$(e_n, e_n) = q^{-(n \choose 2)}$$
Summary of Diagrammatic Rules for any $q$

Theorem

$$(e_n, e_n) = q^{-\binom{n}{2}}$$

Diagrammatics

- There is at most one strand connecting an orange ($h$) platform and a blue ($e$) platform.
- There is a sign as given above when $n$ strands connect two blue platforms.
Relations and the Center

Theorem

$h_1^n$ is in the center of $N\Lambda^q$, if $q^n = 1$. 
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$(h_{1112}, e_4x) = (h_{2111}, e_4x)$
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\( h_1^n \) is in the center of \( N \Lambda^q \), if \( q^n = 1 \).

\((h_{1112}, e_4x) = (h_{2111}, e_4x)\)

\[
\begin{align*}
v_1 &= h_{11211} + h_{12111} + h_{21111} \\
v_2 &= h_{1122} - 2h_{1221} + 3h_{2112} + h_{2211} \\
v_3 &= 2h_{1131} - 2h_{114} + 2h_{1311} - 2h_{141} + 3h_{222} + 2h_{1113} - 2h_{411} \\
v_1 + q^2v_2 + qv_3 &= 0
\end{align*}
\]
$q$-divided Difference Operators

**Definition**

The ring of $q$-symmetric polynomials ($qPol_a$):
\[
\mathbb{Z}\langle x_1, x_2, \ldots, x_a \rangle / \langle x_j x_i - qx_i x_j = 0 \text{ if } j > i \rangle
\]
**q-divided Difference Operators**

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The ring of $q$-symmetric polynomials ($qPol_a$):

$$\mathbb{Z}\langle x_1, x_2, ..., x_a \rangle / \langle x_j x_i - qx_i x_j = 0 \text{ if } j > i \rangle$$

We now define the linear $q$-divided difference operators:

\[ \partial_i (1) = 0 \]
\[ \partial_i (x_i) = q \partial_i (x_{i+1}) = -1 \]
\[ r_i (x_i) = qx_i + 1 \]
\[ r_i (x_{i+1}) = q^{-1} x_i \]
\[ r_i (x_j) = qx_j \text{ if } j > i + 1 \]
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- $\partial_i(x_j) = 0 \text{ if } j \neq i, i + 1$
Definition

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$$r_i(x_i) = q x_{i+1}$$
$$r_i(x_{i+1}) = q^{-1} x_i$$
$$r_i(x_j) = q x_j \text{ if } j > i + 1$$
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\textbf{q-divided Difference Operators}

\textbf{Definition}

The ring of $q$-symmetric polynomials ($q\text{Pol}_a$):
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\end{align*}
\]
\[
\begin{align*}
\partial_i(x_i) &= qx_i+1 \\
\partial_i(x_{i+1}) &= q^{-1}x_i \\
r_i(x_j) &= qx_j \text{ if } j > i + 1 \\
r_i(x_j) &= q^{-1}x_j \text{ if } j < i \\
\end{align*}
\]

Leibniz Rule: $\partial_i(fg) = \partial_i(f)g + r_i(f)\partial_i(g)$
**q-divided Difference Operators**

**Definition**

The ring of q-symmetric polynomials ($qPol_a$):
$$\mathbb{Z}\langle x_1, x_2, ..., x_a \rangle / \langle x_j x_i - q x_i x_j = 0 \text{ if } j > i \rangle$$

We now define the linear $q$-divided difference operators:

- $\partial_i(1) = 0$
- $\partial_i(x_i) = q$
- $\partial_i(x_{i+1}) = -1$
- $\partial_i(x_j) = 0$ if $j \neq i, i + 1$

**Leibniz Rule:**

$$\partial_i(fg) = \partial_i(f)g + r_i(f)\partial_i(g)$$

Note that these definitions account for the odd case as well.
Properties of the $q$-divided Difference Operators

Lemma

$$\partial_i(x_j x_i - q x_i x_j) = 0 \text{ for } j > i.$$
Properties of the $q$-divided Difference Operators

**Lemma**

$$\partial_i(x_jx_i - qx_i x_j) = 0 \text{ for } j > i.$$ 

As a consequence, $\partial_i$ descends to an operator on $q\text{Pol}_{a}$.
Properties of the $q$-divided Difference Operators

Lemma

\[ \partial_i(x_j x_i - q x_i x_j) = 0 \text{ for } j > i. \]

As a consequence, $\partial_i$ descends to an operator on $q\text{Pol}_a$

We have the following properties of the $q$-divided difference operators:
Properties of the $q$-divided Difference Operators

Lemma

$$\partial_i(x_j x_i - qx_i x_j) = 0 \text{ for } j > i.$$  

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We have the following properties of the $q$-divided difference operators:

$$\partial_i^2 = 0$$
Properties of the $q$-divided Difference Operators

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We have the following properties of the $q$-divided difference operators:

- $\partial_i^2 = 0$
- $\partial_i \partial_j = q \partial_j \partial_i$ when $i > j + 1$
Properties of the $q$-divided Difference Operators

Lemma

$$\partial_i(x_j x_i - q x_i x_j) = 0 \text{ for } j > i.$$ 

As a consequence, $\partial_i$ descends to an operator on $q \text{Pol}_a$. We have the following properties of the $q$-divided difference operators:

- $\partial_i^2 = 0$
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\[ \partial_i(x_i^m x_{i+1}^m) = 0 \text{ for any positive integer } m \]
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\[ \partial_i \partial_j = q^{-1} \partial_j \partial_i \text{ when } i < j \]

\[ \partial_i(x^m_i x^m_{i+1}) = 0 \text{ for any positive integer } m \]

\[ \partial_i(x^k_i) = \sum_{j=0}^{k-1} q^{jk-2j-j^2+k} x^j_i x^{k-1-j}_{i+1} \]
Properties of the $q$-divided Difference Operators

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We have the following properties of the $q$-divided difference operators:

- $\partial_i^2 = 0$
- $\partial_i \partial_j = q \partial_j \partial_i$ when $i > j + 1$
- $\partial_i \partial_j = q^{-1} \partial_j \partial_i$ when $i < j$
- $\partial_i(x_i^m x_{i+1}^m) = 0$ for any positive integer $m$

\[ \partial_i(x_i^k) = \sum_{j=0}^{k-1} q^{j(k-2) - j^2 + k} x_i^j x_{i+1}^{k-1-j} \]

\[ \partial_i(x_{i+1}^k) = -\sum_{j=0}^{k-1} q^{-j} x_i^j x_{i+1}^{k-1-j} \]
Properties of the $q$-divided Difference Operators

Definition

Define the $k$-th elementary $q$-symmetric polynomial to be

\[ e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \]

where

\[ \tilde{x}_j = q^{j-1}x_j. \]
Properties of the $q$-divided Difference Operators

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Define the $k$-th elementary $q$-symmetric polynomial to be

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$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_n}$$

and the $k$-th twisted elementary $q$-symmetric polynomial:

$$\tilde{e}_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \tilde{x}_{i_1} \cdots \tilde{x}_{i_n},$$

where $\tilde{x}_j = q^{j-1}x_j$. 
Properties of the $q$-divided Difference Operators

**Theorem**

$$\partial_i(\tilde{e}_k) = 0.$$  
Hence $\tilde{\Lambda}_n^q \subseteq \bigcap_{i=1}^{n-1} \ker(\partial_i)$. 

Properties of the $q$-divided Difference Operators

**Theorem**

\[ \partial_i(\tilde{e}_k) = 0. \]

Hence \( \tilde{\Lambda}_n^q \subseteq \bigcap_{i=1}^{n-1} \ker(\partial_i) \).

**Conjecture**

\[ \bigcap_{i=1}^{n-1} \ker(\partial_i) \subseteq \tilde{\Lambda}_n^q. \]
More properties

nilHecke Relations

\[ \partial_i x_i - qx_{i+1} \partial_i = q \]
\[ \partial_i x_{i+1} - \frac{1}{q} x_i \partial_i = -1 \]
More properties

nilHecke Relations

\[ \partial_i x_i - qx_{i+1} \partial_i = q \]
\[ \partial_i x_{i+1} - \frac{1}{q} x_i \partial_i = -1 \]

Braiding Relation

\[ \partial_i \partial_{i+1} \partial_i \partial_{i+1} \partial_i \partial_{i+1} = -\partial_{i+1} \partial_i \partial_{i+1} \partial_i \partial_{i+1} \partial_i \]
References


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