Higher Bruhat order on Weyl groups of Type B

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Outline

Purpose

For Weyl Groups of Type A, both the Bruhat order and the Higher Bruhat order have a structure that is well understood. Also, for Type B, the Bruhat order is known. However, it is unknown whether the Higher Bruhat order exists for Type B. We claim that this Higher order exists, and has a similar structure and properties as in Type A.

Symmetric Group

- The Symmetric group S_n on the set $1, 2, ..., n$ is the group whose elements are all the permutations of the n symbols.
- Group operation is composition of permutations, treated as bijective functions from the set of symbols to itself.

Diagram

- Element of S_n can be represented by diagrams.
- Ex.: the permutation 3241 is an element of S_4, and it is given by the product of elementary transpositions (23)(34)(23)(12)(34)(23)
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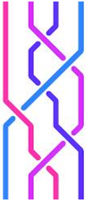
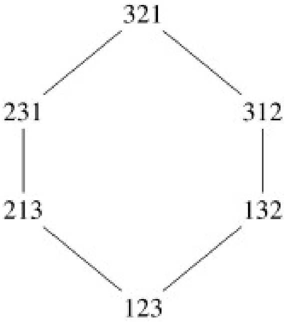
Definitions

- For a permutation u of $1, 2, ..., n$, an inversion is defined as a pair (i, j) such that $1 \leq i < j \leq n$ and $u(i) > u(j)$.
- Inv(u) is the set of all such inversions.
- 
- $Inv(3241) = (12), (14), (24), (34)$

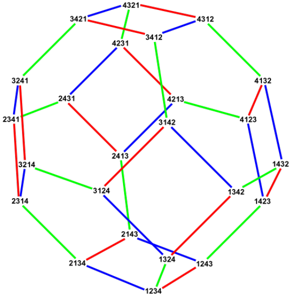
Bruhat Graph

- The packet $P(K)$ of the set $K = i_1, i_2, ..., i_{k+1}$ is the set of $K_{\hat{a}} = K - (i_a)$ for $a = 1, 2, ..., k+1$
- Each edge of the graph is an inversion such as (23)
- Each vertex is an element of the group.
- 

Admissible orders

- $C(n,k)$ denotes the set of all subsets of $(1,2,...,n)$ of cardinality k.
- An admissible total order A(n,k) on C(n,k) is one that induces either lexicographic or antilexicographic order on each packet of size $k+1$.
- An inversion of an admissible order ρ is the packet P(K) such that ρ induces an antilexicographic order on P(K)
- Ex.: (12)(34)(14)(24)(13)(23)

n=4



- Identity and longest permutations appear at opposite vertices.
- Inversions of packets that form chains used to transform lexicographic to antilexicographic.

Higher Bruhat order in type A

- Theorem (Manin-Schechtman): Paths from source to sink can be ordered by inversions.

Types of Inversions

- In Type A, elements of group generated by elementary transpositions (crossings).
- Simiarly, the group of Type B, can be defined as diagrams on 2n strings which are symmetric.

- for C(4, 2), the lexicographic order is (12)<(13)<(14)<(23)<(24)<(34)
- Inversions J_1, J_2 commute if $|J_1 \cup J_2| \geq k + 2$
- Thus, (23) and (14) commute. The new order and lexicographic order said to be elemantarily equivalent.

Type B

- Bruhat order on C(n-1,k) contained in C(n,k)
- $C(3,2) : (-3-2)(-3-1)(-30)(-3+1)(-3+2)(-2-1)(-20)(-2+1)(-10)$

- Define the conjugate pair of inversions with repect to the Type A inversion $J \in C(n-1, k-1)$ as the pair containing $(-n+J)$ and $(-n-J)$.
- $C(3,3) : (-2-1+3)(-3-2-1)(-3-20)(-3-2+1)(-3-1+2)(-3-10)(-2-10)$
- For above Bruhat order, members of conjugate pairs are adjacent.

- Conjecture: Lexicographic order on C(n,k) defined recursively, using conjugate pairs with respect to $J \in C(n-1, k-1)$
- Lexicographic order on Type B gives paths from source to sink, modulo equivalence of commuting inversion, and ordered by inversions through packets.

References

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Images taken from:

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