Minimum Degrees of Minimal Ramsey Graphs

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A graph $F$ is a set of vertices $V$ with an edge set $E \subseteq \binom{V}{2}$.
Write $F \rightarrow H$ to mean that for any two coloring of the edges of $F$, there is a monochromatic copy of $H$.

Removing any edge or vertex of $F$ would make a new graph $F' \nleftrightarrow H$.

A graph which satisfies both of the above properties is said to be a minimal Ramsey graph for $H$.

The family of all minimal Ramsey graphs for $H$ is denoted by $M(H)$. 
Some Classical Problems

- Ramsey’s Theorem: For any $H$, the set $M(H)$ is nonempty.
- Is $M(H)$ finite or infinite?
- Smallest number of edges contained in any graph in $M(H)$.
- The Ramsey number $R(H)$: the smallest number of vertices contained in any graph in $M(H)$.
  - One of the most important, central, and famous problems in combinatorics.
In this presentation, we are concerned with the value

\[ s(H) := \min_{F \in M(H)} \delta(F) \]

\( \delta(F) \) is the minimum degree of the graph \( F \).

In general, this has only been solved for a few classes of graphs.
Example of Finding $s(H)$

Clearly, we must have $s(H) > 0$.
Can we have $s(H) = 1$?

$F' \quad F \in M(H) \quad \delta(F) = 1$
Generally, for any graph $H$ there exists a graph $F'$ such that $F' \nrightarrow H$ and the colors of some portion of $F'$ are fixed however* we want.

How does this help?

We can create whatever* two coloring of a graph we want, then stick a vertex onto this graph.

Argue that no matter which way we color the “stuck on” edges, we will always get a monochromatic copy of $H$. 
Consider $K_5 - edge$ as an example.

Apply the theorem to fix colors and stick a vertex onto it.
In general, take any minimal graph $F$ with $F \rightarrow H$ and remove a vertex $v$ of degree $\delta(F)$.

Take any coloring of $F - v$ without a monochromatic copy of $H$ and see what happens when you put $v$ back in.

$$s(K_t - \text{edge}) = (t - 2)^2$$
For the Future

We proved that \( s(K_{2t} - matching) \leq (t - 1)(2t - 1) \)

\[ s(K_4 - matching) \]

What would be interesting would be to find \( s(G(n, p)) \)