Investigating GCD in Euclidean Domains

Rohil Prasad
under mentorship of Tanya Khovanova

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The greatest common divisor, or GCD, of two integers is the largest integer that divides both of them.

- Many algorithms use GCD calculation, one of the more famous being the RSA encryption algorithm.
- Several algorithms have been devised to efficiently calculate GCD for integers.
The Euclidean Algorithm

def euclid_gcd(a, b):
    if b > a:
        a, b = b, a
    while (!(b divides a)):
        q = a/b
        a, b = b, a-q*b
    return b

(1038, 243)
The Euclidean Algorithm

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(21, 3)
The Binary GCD Algorithm

def binary_gcd(a, b):
    r = 0
    while (a != b):
        while (a, b) are even:
            a, b = a/2, b/2
            r = r+1
        while a is even:
            a = a/2
        while b is even:
            b = b/2
        a, b = min(a, b), abs(a-b)/2
    return a*(2^r)
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Euclidean Domains

We can extend our definition of GCD to arbitrary Euclidean domains.

- A Euclidean domain $E$ is a principal ideal domain with a function $f$ such that for any nonzero $a$ and $b$ in $E$, there exists $q$ and $r$ in $E$ with $a = bq + r$ and $f(r) < f(b)$. This function is called a norm, and $q$ is called the quotient of $a$ and $b$.
- The integers are an example of a Euclidean domain with norm $f(a) = |a|$.
- We work in $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$. 
What are some ways of efficiently calculating GCD in Euclidean domains?

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- Option 3: Approximate division
The quotient of elements in $\mathbb{Z}[\sqrt{2}]$ is calculated as follows:

$$\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = \frac{(a+b\sqrt{2})(c-d\sqrt{2})}{c^2-2d^2} = \frac{ac-2bd}{c^2-2d^2} + \frac{(bc-ad)\sqrt{2}}{c^2-2d^2}$$

Rounding each component to the nearest integer gives the quotient.

Quotient calculation is identical in $\mathbb{Z}[\sqrt{3}]$. 


Option 1: Calculating the quotient

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We use $1 + \sqrt{3}$ for $\mathbb{Z}[\sqrt{3}]$ and $2 \pm \sqrt{2}$ for $\mathbb{Z}[\sqrt{2}]$. 
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Option 3: Approximate Division

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- A possible solution to this is to *approximate* the quotient quickly.
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- A possible solution to this is to approximate the quotient quickly.
- Current implementations for $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$ involves bitshifting the components of each number by about half their bitsize and approximating the quotient with these.
## Comparing Algorithms

<table>
<thead>
<tr>
<th>Component Bitsize</th>
<th>Algorithm Type</th>
<th>Euclidean</th>
<th>Binary</th>
<th>Approx.</th>
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<td>1.45</td>
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<td>500</td>
<td></td>
<td>8.21</td>
<td>15.96</td>
<td>8.65</td>
</tr>
</tbody>
</table>
Future Research

There are many directions in which this research can be taken:

- Extend ideas to $\mathbb{Z}[\sqrt{d}]$ for squarefree $d$. 
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▶ Extend ideas to $\mathbb{Z}[\sqrt{d}]$ for squarefree $d$.
▶ Improve performance of the new ‘binary’ and ‘approximate division’ algorithms.
▶ Find worst cases for the Euclidean algorithm in $\mathbb{Z}[\sqrt{d}]$. 
Acknowledgments

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