Efficient Calculation of Determinants of Symbolic Matrices with Many Variables

Ziv Scully

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Vectors and Volumes
Vectors and Volumes

\[
\text{area} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
Vectors and Volumes

\[
\text{area} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc
\]
Vectors and Volumes
Vectors and Volumes

\[
\begin{pmatrix}
  a \\
  d \\
  g \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  c \\
  f \\
  i \\
\end{pmatrix}
\]

\[
\text{volume} = \det \begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i \\
\end{pmatrix}
\]
Determinants

2 × 2 matrices:

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \]
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$2 \times 2$ matrices:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$3 \times 3$ matrices:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - afh - bdi - ceg$$
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n × n matrices:

$$\det (A) = \sum_{\sigma \in S_n} \left[ \text{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)} \right]$$
Motivation

- Linear systems.
Motivation

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- Calculus.
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- Control theory.
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- Engineering models.
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- Code generation.
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We are interested in matrices with polynomial entries.
Minor Expansion

Naive calculation requires $\Theta(n! n)$ polynomial multiplications.

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = afkp - aflo - agjp + agln + ahjo - ahkn - bekp + belo + bgip - bglm - bhio + bhkm + cejp - celn - cfip + cflm + chin - chjm - dejo + dekn + dfio - dfkm - dgin + dgjm$$
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$$
\begin{vmatrix}
  a & b & c & d \\
  e & f & g & h \\
  i & j & k & l \\
  m & n & o & p \\
\end{vmatrix} = a(fp - flo - gjp + gln + hjo - hkn)
- b(ekp - elo - gip + glm + hio - hkm)
+ c(ejp - eln - fip + flm + hin - hjm)
- d(ejo - ekn - fio + fkm + gin - gjm)
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$$- b(e(kp - lo) - g(ip - lm) + h(io - km))$$
$$+ c(e(jp - ln) - f(ip - lm) + h(in - jm))$$
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$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = \begin{align*} a(f(kp - lo) - g(jp - ln) + h(jo - kn)) \\ - b(e(kp - lo) - g(ip - lm) + h(io - km)) \\ + c(e(jp - ln) - f(ip - lm) + h(in - jm)) \\ - d(e(jo - kn) - f(io - km) + g(in - jm)) \end{align*}$$

Minor expansion requires $\sum_{i=2}^{n} i \binom{n}{i} \in \Theta(2^n \cdot n)$ polynomial multiplications.
Gaussian Elimination

\[
\begin{vmatrix}
  a & b & c & d \\
  0 & f & g & h \\
  0 & 0 & k & l \\
  0 & 0 & 0 & p \\
\end{vmatrix} = afkp
\]
Gaussian Elimination

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  a & b & c & d \\
  0 & f & g & h \\
  0 & 0 & k & l \\
  0 & 0 & 0 & p \\
\end{vmatrix} = afkp
\]
Gaussian Elimination

\[
\begin{vmatrix}
  a & b & c & \ldots \\
  e & f & g & \ldots \\
  \vdots & \vdots & \vdots & \ddots \\
\end{vmatrix}
\]

\[= \text{det } A \]
Gaussian Elimination

\[
\begin{vmatrix}
ad & b & c & \cdots \\
e & f & g & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{vmatrix}
= \det
\begin{vmatrix}
a & b & c & \cdots \\
e - \frac{e}{a}a & f - \frac{e}{a}b & g - \frac{e}{a}c & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{vmatrix}
\]
Gaussian Elimination

$$\det \begin{pmatrix} a & b & c & \cdots \\ e & f & g & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \det \begin{pmatrix} a & b & c & \cdots \\ 0 & f - \frac{e}{a}b & g - \frac{e}{a}c & \cdots \end{pmatrix}$$
Gaussian Elimination

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\begin{vmatrix}
    a & b & c & \cdots \\
    e & f & g & \cdots \\
    : & : & : & \ddots \\
\end{vmatrix}
= \begin{vmatrix}
    a & b & c & \cdots \\
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    : & : & : & \ddots \\
\end{vmatrix}
\]

\[A^{(1)} = A,\]

\[A^{(k+1)}_{i,j} = A^{(k)}_{i,j} - \frac{A^{(k)}_{i,k}}{A^{(k)}_{k,k}} A^{(k)}_{k,j}\]
Gaussian Elimination

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\end{pmatrix}
\]

\[
A^{(1)} = A, \quad A^{(0)}_{0,0} = 1,
\]

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A^{(k+1)}_{i,j} = A^{(k)}_{i,j} - \frac{A^{(k)}_{i,k}}{A^{(k)}_{k,k}} A^{(k)}_{k,j}
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A^{(k+1)}_{i,j} = A^{(k)}_{i,j} \frac{A^{(k)}_{k,k} - A^{(k)}_{i,k} A^{(k)}_{k,j}}{A^{(k-1)}_{k-1,k-1}}
\]
**Gaussian Elimination**

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\]

*Fraction-free Gaussian elimination* requires \(\sum_{i=1}^{n} \Theta(i^2) \in \Theta(n^3)\) polynomial multiplications and divisions.
Comparison

Preservation of “simple” polynomials (e.g., those with few terms):
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Consider an \( n \times n \) matrix with entries of the form \( \sum_{i=1}^{s} a_{i}x_{i} \):
Comparison

Preservation of “simple” polynomials (e.g., those with few terms):

- Minor expansion: Entries are preserved.
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Consider an $n \times n$ matrix with entries of the form $\sum_{i=1}^{s} a_i x_i$:

$$\text{cost ratio} = \frac{\text{cost of ME}}{\text{cost of FFGE}}$$
Row Permutation

Let $p$, $q$ and $r$ be polynomials with $a$, $b$ and $c$ terms, respectively, with no variables in common.
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We want to defer multiplying by polynomials with many terms.
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We want to defer multiplying by polynomials with many terms.

- Absolute value of determinant is invariant under row swaps.
Experiment Setup

Random polynomial matrices:
Experiment Setup

Random polynomial matrices:

- $9 \times 9$. 
Experiment Setup

Random polynomial matrices:
- $9 \times 9$.
- 5 variables.
Experiment Setup

Random polynomial matrices:

- 9 × 9.
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- For various values of $p$: an entry is 0 with probability $p$ and has between one and four terms otherwise. (Any number is equally likely.)
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- Each term is constant or linear in each variable.
Empirical Results

Experiment Setup

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Sort rows $r$ in ascending order based on these scores:
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Sort rows $r$ in ascending order based on these scores:
- Control, no sorting.
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Experiment Setup

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Sort rows $r$ in ascending order based on these scores:
- Control, no sorting.
- Number of nonzero entries in a row, $\sum_{i=1}^{n}(r_i \neq 0)$.
- Total number of terms in a row, $\sum_{i=1}^{n} \text{terms}(r_i)$.
- Product of one more than number of terms for each entry of a row, $\prod_{i=1}^{n}(\text{terms}(r_i) + 1)$. 
Data

Sorting methods:
- Blue: Number of non-zeros
- Red: Sum of nterms($r_i$)
- Green: Product of (n terms($r_i$) + 1)
Data
Further Questions

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Do more experiments!
- Vary other criteria.
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Investigate specific types of matrices:

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