Infinitesimal Cherednik Algebras

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Introduction

• Motivation: Generalization of the basic representation theory of different algebras, including \( \mathfrak{u}_n \) and \( \mathfrak{sp}_{2n} \ltimes A_n \).

• Main objects: Infinitesimal Cherednik algebras \( H_{\alpha}(\mathfrak{gl}_n) \) and \( H_{\alpha}(\mathfrak{sp}_{2n}) \).

• Work from last year:
  1. Computation of entire center for the case \( H_{\alpha}(\mathfrak{gl}_2) \).
  2. Computation of the Shapovalov determinant.

• New results:
  1. Proof of the formula for the first central element of \( H_{\alpha}(\mathfrak{gl}_n) \).
  2. Conjecture regarding the entire center for all \( H_{\alpha}(\mathfrak{gl}_n) \).
  3. Computation of Poisson center for \( H_{\alpha}(\mathfrak{gl}_n) \) and \( H_{\alpha}(\mathfrak{sp}_{2n}) \).
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Definition

Let $V$ be the standard $n$-dimensional column representation of $\mathfrak{gl}_n$, and $V^*$ be the row representation, $\alpha : V \times V^* \to \mathfrak{u} \mathfrak{g} \mathfrak{l}_n$. The **infinitesimal Cherednik algebra** $H_\alpha$ is defined as the quotient of $\mathfrak{u} \mathfrak{g} \mathfrak{l}_n \ltimes T(V \oplus V^*)$ by the relations:

$$[y, x] = \alpha(y, x), \quad [x, x'] = [y, y'] = 0$$

for all $x, x' \in V^*$ and $y, y' \in V$. In addition, $H_\alpha$ must satisfy the PBW property.
Acceptable deformations $\alpha$

- Etingof, Gan, and Ginzburg proved that $\alpha$ is given by $\sum_{j=0}^{k} \alpha_j r_j$ where $r_j$ is the coefficient of $z^j$ in the expansion of
  \[ \text{tr}(x(1 - zA)^{-1} y) \det(1 - zA)^{-1} \]

- We can naturally consider the $\alpha_j$ as coefficients of a polynomial $\alpha(z) = \sum \alpha_j z^j$.

- If $\alpha(z)$ is a linear polynomial, $H_\alpha \cong U(\mathfrak{sl}_{n+1})$. 
Center is Polynomial Algebra

- Let $t_1$, $t_2$, ..., $t_n$ be the generators for the center of $H_0$:
  \[ t_i = \sum_{j=1}^{n} x_j [\beta_i, y_j], \]
  where $\beta_i$ is defined by $\sum_{i=0}^{n} (-1)^i \beta_i z^i = \det(1 - zA)$.

- Tikaradze proved that there exist unique (up to constant) $c_1, c_2, \ldots, c_n \in \mathfrak{z}(\mathfrak{u}g)$ such that
  \[ \mathfrak{z}(H_\alpha) = k[t_1 + c_1, t_2 + c_2, \ldots, t_n + c_n]. \]
A Poisson algebra is a commutative algebra with a *Poisson bracket* that satisfies
\[ \{ab, c\} = b\{a, c\} + a\{b, c\}. \]

*Note that the Lie bracket satisfies* \([ab, c] = a[b, c] + [a, c]b\).*

We can study the Poisson analogue of infinitesimal Cherednik algebras, defined as \(S(\mathfrak{gl}_n) \ltimes S(V \oplus V^*)\) with \(\{a, b\} = [a, b]\) for \(a, b \in \mathfrak{gl}_n \ltimes (V \oplus V^*).\)

The Poisson center consists of elements \(c\) that satisfy \(\{c, a\} = 0\) for all \(a\).

The Poisson bracket approximates its corresponding Lie bracket. Our Goal: to obtain information about the Lie bracket from the Poisson bracket.
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Poisson infinitesimal Cherednik algebra

A Poisson algebra is a commutative algebra with a *Poisson bracket* that satisfies

\[
\{ab, c\} = b\{a, c\} + a\{b, c\}.
\]

*Note that the Lie bracket satisfies* \([ab,c]=a[b,c]+[a,c]b\).\

We can study the Poisson analogue of infinitesimal Cherednik algebras, defined as \(S(\mathfrak{gl}_n) \ltimes S(V \oplus V^*)\) with \(\{a, b\} = [a, b]\) for \(a, b \in \mathfrak{gl}_n \ltimes (V \oplus V^*)\).

The Poisson center consists of elements \(c\) that satisfy \(\{c, a\} = 0\) for all \(a\).

The Poisson bracket approximates its corresponding Lie bracket. **Our Goal: to obtain information about the Lie bracket from the Poisson bracket.**
Computation of Poisson Center

1. Let $g \in \mathfrak{u} \mathfrak{g} \mathfrak{l}_n$. Then, $\{g, y\} = \sum_{i,j=1}^{n} \frac{\partial g}{\partial e_{ij}} \{e_{ij}, y\}$.

2. 
\[
\{t_k, y\} = \sum_{j=1}^{n} \left( \text{Res}_{z=0} \alpha(z^{-1}) \frac{\text{tr}(x_j(1-zA)^{-1}y)}{z \det(1-zA)} dz \right) \{\beta_k, y_j\}.
\]

3. Thus, if $\{t_k + c_k, y\} = 0$,
\[
\sum_{i,j=1}^{n} \frac{\partial c_k}{\partial e_{ij}} \{e_{ij}, y\} = -\sum_{j=1}^{n} \left( \text{Res}_{z=0} \alpha(z^{-1}) \frac{\text{tr}(x_j(1-zA)^{-1}y)}{z \det(1-zA)} dz \right) \{\beta_k, y_j\}.
\]

4. Key idea: since all terms above are $\mathcal{G} \mathcal{L}_n$-invariant and diagonalizable matrices are dense in $\mathfrak{g} \mathfrak{l}_n$, we can assume $A$ is diagonal.
Computation of Poisson Center

1. Let $g \in \mathfrak{gl}_n$. Then, $\{g, y\} = \sum_{i,j=1}^{n} \frac{\partial g}{\partial e_{ij}} \{e_{ij}, y\}$.

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3. Thus, if $\{t_k + c_k, y\} = 0$,

$$\sum_{i,j=1}^{n} \frac{\partial c_k}{\partial e_{ij}} \{e_{ij}, y\} = - \sum_{j=1}^{n} \left( \text{Res}_{z=0} \alpha(z^{-1}) \frac{\text{tr}(x_j(1-zA)^{-1}y)}{z \det(1-zA)}dz \right) \{\beta_k, y_j\}.$$ 

4. Key idea: since all terms above are $GL_n$-invariant and diagonalizable matrices are dense in $\mathfrak{gl}_n$, we can assume $A$ is diagonal.
Computation of Poisson Center

1. Let $g \in \mathfrak{gl}_n$. Then, $\{g, y\} = \sum_{i,j=1}^{n} \frac{\partial g}{\partial e_{ij}} \{e_{ij}, y\}$.

2. 
\[
\{t_k, y\} = \sum_{j=1}^{n} \left( \text{Res}_{z=0} \alpha(z^{-1}) \frac{\text{tr}(x_j (1 - zA)^{-1} y)}{z \det(1 - zA)} dz \right) \{\beta_k, y_j\}.
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4. Key idea: since all terms above are $GL_n$-invariant and diagonalizable matrices are dense in $\mathfrak{gl}_n$, we can assume $A$ is diagonal.
The Poisson Center

Definition
Let $c(t)$ be the generating function of $c_i$: $c(t) = \sum_{i=1}^{n} (-t)^i c_i$.

Theorem
$$c(t) = - \text{Res}_{z=0} \alpha(z^{-1}) \frac{\text{det}(1 - tA)}{\text{det}(1 - zA)} \frac{tz^{-2}}{1 - tz^{-1}} dz.$$
**Examples**

**Case** $c_1$:

$$c_1 = \text{Res}_{z=0} \alpha(z^{-1}) \det(1 - zA)^{-1} z^{-2} dz = \sum_i \alpha_i \text{tr} S^{i+1} A.$$  

**Case** $c_2$:

$$c_2 = \text{Res}_{z=0} \alpha(z^{-1}) \det(1 - zA)^{-1} z^{-2} (\text{tr} A - z^{-1}) \; dz$$  

$$= \sum_i \alpha_i \left( \beta_1 \text{tr} S^{i+1} A - \text{tr} S^{i+2} A \right).$$
Theorem

\[ [\beta_k, y] = \left\{ \sum_{i=0}^{k-1} \binom{k-n}{i+1} \frac{1}{k-n} \beta_{k-i}, y \right\}. \]

Theorem

\[ [\text{tr } S^k A, y] = \left\{ \sum_{i=0}^{k-1} \frac{1}{k+n+1} \binom{k+n+1}{i} (-1)^i \text{tr } S^{k-i} A, y \right\}. \]
Change of basis

- Let $f(z)$ be the polynomial satisfying
  
  $$f(z) - f(z - 1) = \partial^n(z^n\alpha(z)),$$
  and $g(z) = z^{1-n} \frac{1}{\partial^{n-1}} f(z)$.

- Note that if $g(z) = z^{k+1}$,
  
  $$\alpha(z) = \sum_{i=0}^{k-1} \frac{1}{k + n + 1} \binom{k + n + 1}{i} (-1)^i z^{k-i}.$$
Change of basis

Let \( f(z) \) be the polynomial satisfying
\[
f(z) - f(z - 1) = \partial^n(z^n \alpha(z)), \quad \text{and} \quad g(z) = z^{1-n} \frac{1}{\partial_{n-1}} f(z).
\]

Note that if \( g(z) = z^{k+1} \),
\[
\alpha(z) = \sum_{i=0}^{k-1} \frac{1}{k + n + 1} \binom{k + n + 1}{i} (-1)^i z^{k-i}.
\]

Thus,
\[
\left[ \sum g_{j+1} \text{tr} \ S^{j+1} A, y \right] = \left\{ \sum \alpha_j \text{tr} \ S^{j+1} A, y \right\}.
\]
Change of basis

**Conclusion:**

\[
[t_1, y] = \sum_{i=1}^{n} [x_i, y] y_i = \sum_{i=1}^{n} \{x_i, y\} y_i
\]

\[
= -\{\text{Res}_{z=0} \alpha(z^{-1}) \det(1 - zA)^{-1} z^{-2} dz, y\}
\]

\[
= -[\text{Res}_{z=0} g(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz, y].
\]

Hence, \(c_1 = \text{Res}_{z=0} g(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz\).
The General Formula for the Central Element

Conjecture

Let $c(t)$ be the generating function of the $c_i$, i.e., $c(t) = \sum_{i=1}^{n} t^i c_i$. Let

$$h(t, z) = z^{1-n} \left( \frac{2 \sinh \frac{\partial}{2}}{\partial} \right)^{n-1} \frac{1}{1 + tz} \left( \frac{1}{2 \sinh \frac{\partial}{2}} \right)^{n-1} f(z).$$

Then,

$$c(t) = \text{Res}_{z=0} \frac{t \det(1 + tA)}{z \det(1 - zA)} h(t, z^{-1}) dz.$$
Special Cases

**Case** $c_1$:

$$c_1 = \text{Res}_{z=0} g_1(z^{-1}) \det(1 - zA)^{-1} z^{-1} \, dz,$$

where $g_1(z) = \frac{1}{z^{n-1} \partial^{n-1}} f(z)$.

**Case** $c_2$:

$$c_2 = \text{Res}_{z=0} \det(1 - zA)^{-1} \left( g_1(z^{-1}) \text{tr} A - g_2(z^{-1}) \right) z^{-1} \, dz,$$

where $g_2(z) = \frac{1}{z^{n-1} \partial^{n-1}} \left( zf(z) + \frac{n-1}{2 \tanh \frac{\partial}{2}} f(z) \right)$.
Infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$

**Definition**

Let $V$ be the standard $2n$-dimensional representation of $\mathfrak{sp}_{2n}$ with symplectic form $\omega$, and let $\alpha : V \times V \to \mathfrak{usp}_{2n}$. The **infinitesimal Cherednik algebra** $H_\alpha$ is defined as the quotient of $\mathfrak{usp}_{2n} \rtimes T(V)$ by the relations:

$$[x, y] = \alpha(x, y)$$

for all $x, y \in V$. In addition, $H_\alpha$ must satisfy the PBW property.
Etingof, Gan, and Ginzburg proved that $\alpha$ is given by \( \sum_{j=0}^{k} \alpha_j r_{2j} \) where $r_j$ is the coefficient of $z^j$ in the expansion of

\[
\omega(x, (1 - z^2 A^2)^{-1} y) \det(1 - zA)^{-1} = r_0(x, y) + r_2(x, y) z^2 + \cdots.
\]

Note that since $A \in \mathfrak{sp}_{2n}$, the expansion $\det(1 - zA)^{-1}$ only contains even powers of $z$. 

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PBW pairings

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Infinitesimal Cherednik Algebras
Center for $H_\alpha(sp_{2n})$

- Let
  \[ t_i = \sum_{j=1}^{n} [\beta_i, v_j] v_j^*, \]
  where $\beta_i$ is defined by $\sum_{i=1}^{n} \beta_i z^{2i} = \det(1 - zA)$. By $v_j^*$, we mean the element of $V$ that satisfies $\omega(v_i, v_j^*) = \delta_{ij}$.

- These $t_i$ generate the center of $H_0(sp_{2n})$.

- We conjecture that $\mathcal{z}(H_\alpha) = k[t_1 + c_1, \ldots, t_n + c_n]$, where $c_i \in \mathcal{z}(Usp_{2n})$ are unique up to a constant.
Theorem

Let $c(t)$ be the generating function for the $c_i$:

$$c(t) = \sum_{i=1}^{n} (-1)^{i+1} c_i z^{2i}.$$ 

Then,

$$c(t) = 2 \text{Res}_{z=0} \alpha(z^{-1}) \frac{\det(1 - tA)}{\det(1 - zA)} \frac{z^{-3}}{1 - z^2 t^{-2}} dz.$$
Summary

- Last year, we studied $H_\alpha(\mathfrak{gl}_n)$. We conjectured the formula for the first central element’s action on the Verma module.
- Using this conjecture, we calculated the Shapovalov form and determined the finite dimensional irreducible representations.
- We proved this conjecture earlier this year, thereby proving all aforementioned results.
- Currently, we are trying to find the other central elements. To do this, we pass to the Poisson algebra, which is easier to handle.
- Also, we are trying to extend results from $H_\alpha(\mathfrak{gl}_n)$ to $H_\alpha(\mathfrak{sp}_{2n})$, such as Kostant’s theorem and the classification of finite dimensional representations.
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