Modular representations of Cherednik algebras

Sheela Devadas and Steven Sam

Second Annual MIT PRIMES Conference, May 19, 2012
Representations of Cherednik Algebras

- The Cherednik algebra $H_{\hbar,c}(G, \mathfrak{h})$ is a $\mathbb{Z}$-graded algebra.
- We study the Cherednik algebra in positive characteristic.
- The representations of the algebra we study are constructed from Verma modules $M_c(\tau)$ where $\tau$ is a representation of the group $G$.
- $M_c(\tau)$ is equivalent to $\text{Sym}(\mathfrak{h}^*) \otimes \tau$.
- We construct a submodule $J_c(\tau)$ as the kernel of a bilinear form $\beta_c$ which can be calculated with a computer: the lowest-weight representations of the Cherednik algebra are then $M_c(\tau)/J_c(\tau) = L_c(\tau)$.
- The Hilbert series of $L_c$ is $\sum_{i=0}^{\infty} (\dim(L_c)_i) t^i$.
- The main goal of the project is to be able to compute Hilbert series for all $L_c(\tau)$. We also study the free resolutions of some $L_c(\tau)$, allowing us to approximate certain modules with better-behaved ones.
• Latour, Katrina Evtimova, Emanuel Stoica, Martina Balagovic and Harrison Chen studied the Cherednik algebra for other groups
• Unlike them, we work with groups that are examples in char. 0 reduced mod $p$ and higher rank
• We work with groups $G(m, r, n)$, which are $n$ by $n$ permutation matrices with entries that are $m^{th}$ roots of unity such that the product of the entries is an $\frac{m}{r}^{th}$ root of unity
• With Carl Lian, we were able to find Hilbert series for the groups $G(1, 1, n)$ or $S_n$ when $\hbar = 1$ for some special values of the parameter $c$ for trivial $\tau$: in general, we use generic $c$
• In the case when $G = S_n$, $p$ divides $n$, $\tau$ is trivial, we were able to find Hilbert series for $L_c(\tau)$ and generators for $J_c(\tau)$ for $\hbar = 0$ and for $\hbar = 1, p = 2$
• For $G(m, m, 2)$ and $\hbar = 1$, we were able to find Hilbert series for $L_c(\tau)$ and generators for $J_c(\tau)$ for some $\tau$
\[ \mathcal{h} = 0, \; G(m,m,n) \; \text{and} \; G(m,1,n) \]

- The ideal \( J_c \) has behavior related to subspace arrangements in the case \( G = G(m,1,n) \), which includes the case \( G = S_n \) \((m = 1)\)
- Let \( X_i \) be the set of all \((x_1, \ldots, x_n)\) such that some \( n - i \) of the coordinates are equal.
- Let \( I_{i(m)} \) be the ideal of \( X_i \) in degree \( m \)
- For \( n \equiv i \mod p \) with \( 0 \leq i \leq p - 1 \) and \( \mathcal{h} = 0 \), the data suggests that \( J_c \) is generated by symmetric functions and \( I_{i(m)} \). \( L_c \) seems to be a complete intersection in \( X_i \).
- For \( G(m,m,n) \) we see coordinate subspaces and the related ideals in the behavior of \( J_c \)
- We conjecture that when \( n \equiv 0 \mod p \), the regular sequence is \( x_1^m + \cdots x_n^m, x_1^{2m} + \cdots x_n^{2m}, \ldots, x_1^{(i-1)m} + \cdots x_n^{(i-1)m} \)
- The exception is when \( n \equiv 0 \mod p \), where \( J_c \) is generated by the squarefree monomials of degree \( p \) and the differences of the \( m^{th} \) powers of the \( x_i \)
Dihedral groups $G(m,m,2), \hbar = 0$

- Dihedral groups are the groups $G(m,m,2)$, they can also be considered the group of symmetries of a regular $m$-gon.
- Representations of the dihedral group take the form $\rho_i$ for $0 \leq i < \frac{m}{2}$: these representations are equivalent to the standard 2-dimensional one, except roots of unity act by their $i^{th}$ power (except for $i = 0$, which is the trivial representation).
- There are 1 or 3 additional representations based on tensoring the trivial representation by a character (for example, the sign representation), depending on the parity of $m$.
- These are indexed by negative integers.
- We use these representations as $\tau$. 
Dihedral groups results

- For $i \leq 0$, $\rho_i$ has one basis vector $e_1$; for $i > 0$, $\rho_i$ has two basis vectors $e_1, e_2$

- Let $x_1$ and $x_2$ be basis vectors of $\mathfrak{h}^*$

- The results in this case appear to be independent of characteristic

- If $i \leq 0$, then $x_1 \ast x_2 \otimes e_1, (x_1^m + x_2^m) \otimes e_1$ generate $J_c$

- If $i = 1$, then $x_1 \otimes e_1, x_1^3 \otimes e_2, x_2^3 \otimes e_1, x_2 \otimes e_2$ generate $J_c$

- If $1 < i < \frac{m}{2}$, then $x_1 \otimes e_1, x_1 \otimes e_2, x_2 \otimes e_1, x_2 \otimes e_2$ generate $J_c$ unless $m$ is even and $i = \frac{m}{2} - 1$

- If $i = \frac{m}{2} - 1$ and $m$ is even, then
  $x_1 \otimes e_1, x_1^3 \otimes e_2, x_2^3 \otimes e_1, x_2 \otimes e_2$ generate $J_c$

- $m = 4$ is a special case since $1 = \frac{m}{2} - 1$
Dihedral group free resolutions

- Free resolutions can be calculated for $L_c(\rho_i)$ in most cases (let $A = \text{Sym}(\mathfrak{h}^*)$)

- If $i \leq 0$, then the free resolution is:

  $$0 \leftarrow L_c(\rho_i) \leftarrow \rho_i \otimes A \leftarrow \rho_i \otimes A(-2) \oplus \rho_i \otimes A(-m) \leftarrow \rho_i \otimes A(-m-2) \leftarrow 0$$

- If $i = 1$ the free resolution is:

  $$0 \leftarrow L_c(\rho_1) \leftarrow \rho_1 \otimes A \leftarrow \rho_2 \otimes A(-1) \oplus \rho_2 \otimes A(-3) \leftarrow \rho_1 \otimes A(-4) \leftarrow 0$$
Dihedral group free resolutions

• If $1 < i < \frac{m}{2}$ (unless $m$ is even and $i = \frac{m}{2} - 1$) the free resolution is:

$$
0 \leftarrow L_c(\rho_i) \leftarrow \rho_i \otimes A \leftarrow \rho_i \otimes h^* \otimes A(-1)
$$

$$
\leftarrow \rho_i \otimes \wedge^2 h^* \otimes A(-2) \leftarrow 0
$$

• If $i = \frac{m}{2} - 1$, and $m$ is even and greater than 8, the free resolution is:

$$
0 \leftarrow L_c(\rho_i) \leftarrow \rho_i \otimes A \leftarrow (\rho_{-2} \oplus \rho_{-1}) \otimes A(-1) \oplus \rho_{\frac{m}{2}-4} \otimes A(-3)
$$

$$
\leftarrow \rho_{\frac{m}{2}-3} \otimes A(-4) \leftarrow 0
$$
The following transition matrix, for the case $G(5, 5, 2)$, expresses the characters of the $L_c(\tau)$ as alternating sums of the characters of the Verma modules $M_c(\tau)$, using the variable $t$ to represent grading shifts:

$$
\begin{pmatrix}
(1 - t^2)(1 - t^5) & 0 & 0 & 0 \\
0 & (1 - t^2)(1 - t^5) & 0 & 0 \\
0 & 0 & 1 + t^4 & -t \\
0 & 0 & -t - t^3 & 1 - t + t^2
\end{pmatrix}
$$

The columns of this matrix represent $L_c(\tau)$ for the four representations of $G(5, 5, 2)$, while the rows represent $M_c(\tau)$ for the same four representations (in the order $\rho_{-1}, \rho_0, \rho_1, \rho_2$).
The inverse matrix shows the characters of the $M_c(\tau)$ in terms of the characters of the $L_c(\tau)$, with the fractional coefficient representing that the $L_c(\tau)$ are being infinitely summed. The baby Verma modules $M'_c(\tau)$ are equivalent to $M_c(\tau)$ quotiented by the invariants, which have degrees 2 and 5 for $G(5, 5, 2)$, so when we remove the fractional coefficient, the transitional matrix relates the baby Verma modules to the $L_c(\tau)$. (Here the columns refer to the $M_c(\tau)$ and the rows to the $L_c(\tau)$, with the same indexing of representations.)

$$\frac{1}{(1-t^2)(1-t^5)} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 + t^3 & t + t^2 + t^3 + t^4 \\
0 & 0 & t + t^2 & 1 + t + t^4 + t^5
\end{pmatrix}$$
G(m,m,3) conjectures

- Let $x, y, z$ be basis vectors of $\mathfrak{h}^*$
- $G(m, m, 3)$ has one two-dimensional representation $\gamma_0$: it is equivalent to the standard three-dimensional representation with roots of unity acting trivially, quotiented by the sum of the variables, and it has two basis vectors $e_1$ and $e_2$
- The following results are true when $p > 2$
- In this case we conjecture that $J_c$ is generated by
  $$(x^m + y^m + z^m) \otimes e_1, (x^m + y^m + z^m) \otimes e_2, xyz \otimes e_1, xyz \otimes e_2, -x^m \otimes e_1 + z^m \otimes e_2, y^m \otimes e_1 + -x^m \otimes e_2$$
- $G(m, m, 3)$ has $m - 1$ three-dimensional representations $\gamma_i$ for $1 \leq i \leq m - 1$ equivalent to the standard three-dimensional representation, with roots of unity acting by their $i^{th}$ power (three basis vectors $e_1, e_2, e_3$)
- In this case (unless $i = 1, p = 2, $ or $m = 2$) we conjecture that $J_c$ is generated by
  $$x \otimes e_1, y \otimes e_2, z \otimes e_3, yz \otimes e_1, xz \otimes e_2, xy \otimes e_3, y^{m-i} \otimes e_1 + x^{m-i} \otimes e_2, z^{m-i} \otimes e_3, z^{m-i} \otimes e_2 + y^{m-i} \otimes e_3$$
Further research

• We plan to find the expressions of the $M'_c(\tau)$ in terms of the $L_c(\tau)$ for the remaining cases for the dihedral group and the groups $G(m, m, 3)$ as well

• We also plan to find free resolutions for small cases of $G(m, r, n)$ and use $K$-theory in a similar way
Thanks to PRIMES for providing this opportunity, Pavel Etingof for suggesting this problem, and our mentor Steven Sam.