Degrees of Regularity of Colorings of the Integers

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Original problem: For a given homogeneous equation, how many colors can be used to color the nonzero integers, or the appropriate ring, so that all such colorings give a monochromatic solution?
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Background

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- (Schur, 1916) Schur’s theorem
- (van der Waerden, 1927) van der Waerden’s theorem
- (Rado, 1933) Rado’s theorem
We consider the general equation

\[ a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = n \]
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The equation is *regular under a coloring* if there is a solution \( x_1, x_2, \ldots, x_n \) in which \( x_1, x_2, \ldots, x_n \) all have the same color. Such a solution is said to be monochromatic.
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For example,

\[ x + y = 2z \]

is regular under every coloring.
Definitions and Examples

The equation is $m$-regular if for all colorings $c$ with $m$ colors, the equation is regular under $c$.

Intuitively, the smaller the value of $m$, the more likely it is for an equation to be $m$-regular.

The equation is regular if it is $m$-regular for all $m$.

Examples:

$x + y = z$ is regular: 1 2 3 4 5

$x + 2y = 4z$ is 2-regular, but not 3-regular.

1 2 3 4 5 6 7 8 9 . . .
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Examples:
- \( x + y = z \) is regular: 1 2 3 4 5/5
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- Thus, every equation is 1-regular.
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- *x + y = z* is regular: 1 2 3 4 5/5
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  1 2 3 4 5 6 7 8 9 ...
More Examples

- The equation $x_1 + x_2 + x_3 = 4x_4$ is 3-regular but not 4-regular.

  \[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ \ldots\]
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In fact, there are infinitely many colorings with no monochromatic solutions.
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- The equation $x_1 + x_2 + x_3 = 4x_4$ is 3-regular but not 4-regular.

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  In fact, there are infinitely many colorings with no monochromatic solutions.

- The equation $x_1 + 2x_2 + 3x_3 - 5x_4 = 0$ is completely regular.
Goals

- Determine degrees of regularity for various other equations
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- Characterize equations that are regular under certain colorings
Lemma

If an equation is regular under the periodic coloring with period $p$ and $p$ distinct colors, it is regular under all colorings of period $p$. 
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- This behavior is expected: replace does not change regularity, especially reducing the number of colors.
Lemma

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Idea: Find a solution, and since the equation is homogeneous, multiply everything by $p$. 
Shift non-homogeneous equation: \( y_i = x_i + \gamma. \)

\[
a_1 y_1 + a_2 y_2 + \cdots + a_k y_k = n + S \gamma,
\]

where \( S \) is the sum of the coefficients: \( S = a_1 + a_2 + \cdots + a_k. \)
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$$a_1 y_1 + a_2 y_2 + \cdots + a_k y_k = n + S \gamma,$$

where $S$ is the sum of the coefficients: $S = a_1 + a_2 + \cdots + a_k$.

**Lemma**

*With respect to periodic colorings this equation is equivalent to the main equation.*
Theorem

The main equation is regular under the coloring of period $p$ with $p$ distinct colors if and only if there exists $\gamma$ such that $n \equiv S\gamma \pmod{p}$.
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The main equation is regular if and only if \( S \) divides \( n \).
For any periodic coloring of period \( p \), the main equation has a monochromatic solution if and only if there is one in \( \mathbb{Z}/p\mathbb{Z} \).
Discussion

For any periodic coloring of period $p$, the main equation has a monochromatic solution if and only if there is one in $\mathbb{Z}/p\mathbb{Z}$.

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- This suggests an analogous result for other algebraic structures colored by equivalence class.
- This also presents an inexact criterion for regularity under a periodic coloring with less than $p$ colors.

**Conjecture**

*The main equation is regular under any binary periodic coloring of period $p > 2$.***
We now consider the degree of regularity of the general equation.

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**Lemma**

If \( k \) divides \( S \) and does not divide \( n \), then the equation is not \( k \)-regular.
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**Lemma**

If \( k \) divides \( S \) and does not divide \( n \), then the equation is not \( k \)-regular.

This gives a measure of how far from regular the equation is.
It turns out that this is often not strong at all.
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**Lemma**

*When \(a_1, a_2, a_3, \ldots\) have the same sign, the equation is not 2-regular when \(S\) does not divide \(n\).*
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**Lemma**

*When \(a_1, a_2, a_3, \ldots\) have the same sign, the equation is not 2-regular when \(S\) does not divide \(n\).*

**Conjecture**

*The coloring used in the proof of the previous lemma is the only one that breaks regularity for a binary coloring.*
Future Research

- Work with other types of colorings, e.g. equidistributed colorings, example colorings shown earlier.
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- Find degrees of regularity for specific equations, e.g. \( ax + by = z \) for various \( a, b, x^2 + y^2 = z^2 \).
- Find some structure on colorings and/or equations. A basic example of this was the shifting property for periodic colorings.
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