Beyond Alternating Permutations: Pattern Avoidance in Young Diagrams and Tableaux

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A permutation \( w \) is called *alternating* if

\[
w_1 < w_2 > w_3 < w_4 > \cdots .
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For example, 352614 is alternating. Graphically, this is
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Given a permutation $q$ and a positive integer $n$, let $S_n(q)$ ($A_n(q)$) denote the set of all (alternating) permutations of length $n$ that avoid $q$. 
Pattern Containment in Permutations

- A permutation \( w \) is said to contain a permutation \( q \) if there is a subsequence of \( w \) order-isomorphic to \( q \). If \( w \) does not contain \( q \), then \( w \) avoids \( q \).

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- If \( |S_n(p)| = |S_n(q)| \) for all \( n \), we say that \( p \) and \( q \) are Wilf-equivalent.
A permutation $w$ is said to contain a permutation $q$ if there is a subsequence of $w$ order-isomorphic to $q$. If $w$ does not contain $q$, then $w$ avoids $q$. For example, $325641$ contains $231$.

Given a permutation $q$ and a positive integer $n$, let $S_n(q)$ ($A_n(q)$) denote the set of all (alternating) permutations of length $n$ that avoid $q$.

If $|S_n(p)| = |S_n(q)|$ for all $n$, we say that $p$ and $q$ are Wilf-equivalent.

If $|A_n(p)| = |A_n(q)|$ for all $n$, we say that $p$ and $q$ are equivalent for alternating permutations.
(Mansour, Deutsch, Reifegerste) If $q$ is a pattern of length 3, then $|A_n(q)|$ is a Catalan number (i.e. of the form $C_k = \frac{(2k)!}{k!(k+1)!}$).

The indices depend on the choice of $q$ and on the parity of $n$. 
Previous Results

- **(Mansour, Deutsch, Reifegerste)** If \( q \) is a pattern of length 3, then \( |A_n(q)| \) is a Catalan number (i.e. of the form \( C_k = \frac{(2k)!}{k!(k+1)!} \)).

- **(Lewis)** For patterns of length 4,

\[
|A_{2n}(1234)| = |A_{2n}(2143)| = \frac{2(3n)!}{n!(n+1)!(n+2)!},
\]

\[
|A_{2n+1}(1234)| = \frac{16(3n)!}{(n-1)!(n+1)!(n+3)!},
\]

\[
|A_{2n+1}(2143)| = \frac{2(3n+3)!}{n!(n+1)!(n+2)!(2n+1)(2n+2)(2n+3)}.
\]
**The Main Theorem and Its Motivation**

**Theorem** (Backelin-West-Xin). For all $t \geq k$ and all permutations $q$ of $\{k + 1, k + 2, k + 3, \ldots, t\}$, the patterns $123 \cdots kq$ and $k(k-1)(k-2)\cdots1q$ are Wilf-Equivalent.
The Main Theorem and Its Motivation

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**Main Results:**
For all $q$, the following sets of patterns are equivalent for alternating permutations.

- $12q$ and $21q$
The Main Theorem and Its Motivation

**Theorem** (Backelin-West-Xin). *For all* \( t \geq k \) *and all permutations* \( q \) *of* \( \{k + 1, k + 2, k + 3, \cdots, t\} \), *the patterns* \( 123 \cdots kq \) *and* \( k(k-1)(k-2)\cdots 1q \) *are Wilf-Equivalent.*

**Main Results:**
For all \( q \), the following sets of patterns are equivalent for alternating permutations.

- \( 12q \) and \( 21q \)
- \( 123q, 213q \) and \( 321q \)
The Main Theorem and Its Motivation

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**Main Results:**
For all \( q \), the following sets of patterns are equivalent for alternating permutations.

- \( 12q \) and \( 21q \)
- \( 123q, 213q \) and \( 321q \)
- (Conjecture) For all \( k \), \( 123 \cdots kq \) and \( k(k-1)(k-2)\cdots1q \)
Pattern Avoidance of Young Diagrams

Basic Definitions
Ascents/Descents
Alternation
Permutations
Matrix Extension
Main Theorem
Beyond Alternating Permutations
A **Young diagram** with $n$ rows/columns is a set $Y$ of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of $S$ is also in $Y$. 
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A *transversal* $T$ of $Y$ is a set of squares of $Y$ that contains exactly one member per row and per column of $Y$. 

\[\text{Diagram showing a transversal of a Young diagram.}\]
Basic Definitions

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- $T$ is said to **contain** a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are $k$ rows $r_1 < r_2 < \cdots < r_k$ and $k$ columns $c_1 < c_2 < \cdots < c_k$ of $Y$ such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$.

contains \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
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\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

contains

All 4 red squares are in the Young diagram.
**Basic Definitions**

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is not a copy of \[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \].
A **Young diagram** with \( n \) rows/columns is a set \( Y \) of squares of an \( n \times n \) board such that if a square \( S \in Y \), then any square above and to the left of \( S \) is also in \( Y \).

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The square \( X \) is not in the Young diagram.
Basic Definitions

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Basic Definitions

- A *Young diagram* with $n$ rows/columns is a set $Y$ of squares of an $n \times n$ board such that if a square $S \in Y$, then any square above and to the left of $S$ is also in $Y$.
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- $T$ is said to *contain* a $k \times k$ permutation matrix $M = (m_{i,j})$ if there are $k$ rows $r_1 < r_2 < \cdots < r_k$ and $k$ columns $c_1 < c_2 < \cdots < c_k$ of $Y$ such that $(r_k, c_k) \in Y$ and $(r_i, c_j) \in T$ if and only if the entry of $m_{i,j} = 1$. Otherwise, we say that $T$ *avoids* $M$.
- If permutation matrices $M$ and $M'$ are such that, for all Young diagrams $Y$, the number of transversals of $Y$ avoiding $M$ is the same as the number avoiding $M'$, we say that $M$ and $M'$ are *shape-Wilf equivalent*. 
Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that $i$ is *an ascent of $T$* (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$. 
Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that $i$ is an ascent of $T$ (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.

An AD-Young diagram is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram $Y$ with $n$ rows, and disjoint sets $A, D \subseteq [n - 1]$ such that if $i \in A \cup D$, then the $i$th and $(i + 1)$st rows of $Y$ have the same length.
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$$Y = \begin{array}{cccccccc}
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} & \text{\cellcolor{yellow}} \\
\end{array}$$

$A = \{1\}$  \hspace{1cm} $D = \{3\}$
Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that $i$ is an ascent of $T$ (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.

An AD-Young diagram is a triple $\mathcal{Y} = (Y, A, D)$ of a Young diagram $Y$ with $n$ rows, and disjoint sets $A, D \subseteq [n - 1]$ such that if $i \in A \cup D$, then the $i$th and $(i + 1)$st rows of $Y$ have the same length.

A valid transversal of $\mathcal{Y}$ is a transversal $T$ of $Y$ such that if $i \in A (D)$, then $i$ is an ascent (descent) of $T$.

$Y = \begin{array}{cccccccc}
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} & \text{Y} \\
\end{array}$

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Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that $i$ is an ascent of $T$ (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.

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\[
\begin{array}{c}
Y = \\
A = \{1\} \\
D = \{3\}
\end{array}
\]
Given a transversal $T = \{(i, b_i)\}$ of a Young diagram, we say that $i$ is an ascent of $T$ (descent) when it is an ascent (descent) of $b_1 b_2 \cdots$.

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A valid transversal of $\mathcal{Y}$ is a transversal $T$ of $Y$ such that if $i \in A \ (D)$, then $i$ is an ascent (descent) of $T$. Pattern avoidance is exactly as in Young diagrams.
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A valid transversal of $\mathcal{Y}$ is a transversal $T$ of $Y$ such that if $i \in A$ ($D$), then $i$ is an ascent (descent) of $T$. Pattern avoidance is exactly as in Young diagrams.

Given a permutation matrix $M$ and an AD-Young diagram $\mathcal{Y}$, let $S_{\mathcal{Y}}(M)$ denote the set of valid transversals of $\mathcal{Y}$ that avoid $M$. 

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**Pattern Avoidance in Alternating Permutations**

**Pattern Avoidance of Young Diagrams**

**Basic Definitions**

**Ascents/Descents**

**Alternation Permutations**

**Matrix Extension**

**Main Theorem**

**Beyond Alternating Permutations**
An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with $Y$ a Young diagram with $n$ columns is called $x$-alternating if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$. 

$$Y = \begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \\
\bullet & \\
\end{array}$$
An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with $Y$ a Young diagram with $n$ columns is called $x$-alternating if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.
An AD-Young diagram \( \mathcal{Y} = (Y, A, D) \) with \( Y \) a Young diagram with \( n \) columns is called \( x \)-alternating if it satisfies the property that if \( i \leq n - x \), then \( i \in A \) if and only if \( i + 1 \in D \).

\[
\begin{align*}
Y &= \\
A &= \{1\} \\
D &= \{2\}
\end{align*}
\]

is 1-alternating.
An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with $Y$ a Young diagram with $n$ columns is called $x$-alternating if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$. 

$$Y = \begin{array}{cccccc}
\end{array}$$
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$Y = \begin{array}{cccccccc}
\text{ Yellow boxes } \\
\end{array}$

$A = \{2\}$

$D = \emptyset$
An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with $Y$ a Young diagram with $n$ columns is called $x$-alternating if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

$Y =$

$A = \{2\}$

$D = \emptyset$

is 4-alternating.
Alternating AD-Young Diagrams

- An AD-Young diagram $\mathcal{Y} = (Y, A, D)$ with $Y$ a Young diagram with $n$ columns is called $x$-alternating if it satisfies the property that if $i \leq n - x$, then $i \in A$ if and only if $i + 1 \in D$.

- If $M$ and $M'$ are permutation matrices such that for all $x$-alternating AD-Young diagrams $\mathcal{Y}$, we have $|S_{\mathcal{Y}}(M)| = |S_{\mathcal{Y}}(M')|$, then we say that $M$ and $M'$ are shape-equivalent for $x$-alternating AD-Young diagrams.
We can treat a permutation $b$ of length $n$ as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.
We can treat a permutation $b$ of length $n$ as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.

We can treat an alternating permutation $b$ of length $2n$ as a valid transversal $\{(i, b_i)\}$ of the 2-alternating AD-Young diagram $(Y, A, D)$ with $Y$ the $2n \times 2n$ square, $A = \{1, 3, 5, \ldots, 2n-1\}$, and $D = \{2, 4, 6, \ldots, 2n-2\}$. The permutation 352614 is
We can treat a permutation $b$ of length $n$ as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.

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A permutation $b$ avoids a pattern $q$ if and only if its corresponding transversal avoids $q$'s permutation matrix.
Alternating Permutations as Transversals

- We can treat a permutation $b$ of length $n$ as a transversal $\{(i, b_i)\}$ of the $n \times n$ Young diagram.

- We can treat an alternating permutation $b$ of length $2n$ as a valid transversal $\{(i, b_i)\}$ of the 2-alternating AD-Young diagram $(Y, A, D)$ with $Y$ the $2n \times 2n$ square, $A = \{1, 3, 5, \cdots, 2n - 1\}$, and $D = \{2, 4, 6, \cdots, 2n - 2\}$.

- A permutation $b$ avoids a pattern $q$ if and only if its corresponding transversal avoids $q$’s permutation matrix.

- Similarly, alternating permutations of odd length, can be treated as valid transversals of 1-alternating AD-Young diagrams.
Theorem (Babson-West). If $M$ and $M'$ are permutation matrices that are shape-Wilf equivalent, and $P$ is an permutation matrix of positive dimensions, then the matrices

$$\begin{bmatrix} M & 0 \\ 0 & P \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M' & 0 \\ 0 & P \end{bmatrix}$$

are shape-Wilf equivalent.
Theorem (Babson-West). If \( M \) and \( M' \) are permutation matrices that are shape-Wilf equivalent, and \( P \) is an permutation matrix of positive dimensions, then the matrices

\[
\begin{bmatrix}
M & 0 \\
0 & P
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
M' & 0 \\
0 & P
\end{bmatrix}
\]

are shape-Wilf equivalent.

Theorem. If \( M \) and \( M' \) are permutation matrices that are shape-Equivalent for \( x \)-alternating AD-Young diagrams, and \( P \) is an \( r \times r \) permutation matrix, then the matrices

\[
\begin{bmatrix}
M & 0 \\
0 & P
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
M' & 0 \\
0 & P
\end{bmatrix}
\]

are shape-equivalent for \( x + r \)-alternating AD-Young diagrams.
Theorem (Backelin-West-Xin). For all $k$, the permutation matrices of the permutations $(k-1)(k-2)(k-3)\cdots 1k$ and $k(k-1)(k-2)\cdots 1$ are shape-Wilf equivalent.
The Main Theorem Revisited

**Theorem** (Backelin-West-Xin). For all $k$, the permutation matrices of the permutations $(k - 1)(k - 2)(k - 3)\cdots 1k$ and $k(k - 1)(k - 2)\cdots 1$ are shape-Wilf equivalent.

**Theorem.** The permutation matrices corresponding to the permutations 12 and 21 are shape-equivalent for 1-alternating AD-Young diagrams.
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**Theorem.** The permutation matrices corresponding to the permutations 213 and 321 are shape-equivalent for 1-alternating AD-Young diagrams.
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**Theorem** (Backelin-West-Xin). For all \( k \), the permutation matrices of the permutations \((k - 1)(k - 2)(k - 3) \cdots 1k\) and \(k(k - 1)(k - 2) \cdots 1\) are shape-Wilf equivalent.

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**Theorem.** The permutation matrices corresponding to the permutations 213 and 321 are shape-equivalent for 1-alternating AD-Young diagrams.

**Corollary.** For all \( t > 2 \) and all permutations \( q \) of \( \{3, 4, 5, \cdots, t\} \), the patterns 12\( q \) and 21\( q \) are equivalent for alternating permutations. For all \( t > 3 \) and all permutations \( q \) of \( \{4, 5, 6, \cdots, t\} \), the patterns 123\( q \), 213\( q \) and 321\( q \) are equivalent for alternating permutations.
Beyond Alternating Permutations

Pattern Avoidance in Alternating Permutations
Pattern Avoidance of Young Diagrams

Motivation
Reading Words
321 Avoidance
Proof
321 Applications
Data for \( l = 0 \)
Investigating \( l = 0 \)
Repetitive Patterns
Further Work
Motivation

- Joel’s question in his paper.
Motivation

- Joel’s question in his paper.
- Bijection from permutations to Young tableaux
  - Definition of tableau
  - Entries increase left to right; top to bottom
Motivation

- Joel’s question in his paper.
- Bijection from permutations to Young tableaux

**Definition of tableau**

- Entries increase left to right; top to bottom
- $l$: Number of adjacent edges between adjacent rows
- $k$: Number of cells per row (except top row)
- $n$: Total number of cells/values in the permutation
- Ex. $(2, 4, 10)$; $l = 2, k = 4, n = 10$
Reading word: $124(10)357968$

Pattern avoidance is exactly as in permutations.
- Reading word: 124(10)357968
- Pattern avoidance is exactly as in permutations.
- Define $U_{n}^{k,l}(r)$ to be the set of permutations $p$ that fill tableau of the form $(l, k, n)$ and such that $p$ avoids $r$. 
Pattern avoidance in alternating permutations

Pattern avoidance of Young diagrams

Beyond alternating permutations

Motivation

Pattern avoidance is exactly as in permutations.

Define $U_n^{k,l}(r)$ to be the set of permutations $p$ that fill tableau of the form $(l, k, n)$ and such that $p$ avoids $r$.

Alternating permutation pattern avoidance is a special case: $A_n(r) = U_n^{2,1}(r)$. 

Reading word: 124(10)357968

Reading Words

321 Avoidance

Proof

321 Applications

Data for $l = 0$

Investigating $l = 0$

Repetitive Patterns

Further Work
Theorem. For $t > 1$, we have

$$\left| U_{kt+1}^{k,1}(321) \right| = \sum_{i=k(t-1)+2}^{kt} \left| U_{i}^{k,1}(321) \right|.$$
Theorem. For \( t > 1 \), we have

\[
\left| U_{kt+1}^{k,1}(321) \right| = \sum_{i=k(t-1)+2}^{kt} \left| U_{i}^{k,1}(321) \right|.
\]

Example when \( k = 3 \):

\[
\left| U_{3t+1}^{3,1}(321) \right| = \left| U_{3t-1}^{3,1}(321) \right| + \left| U_{3t}^{3,1}(321) \right|
\]

Some data:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_{n}^{3,1}(321) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>19</td>
<td>28</td>
<td>90</td>
<td>207</td>
<td>297</td>
</tr>
</tbody>
</table>
$l = 1$: one edge shared between adjacent rows

\[ a_{kt} \quad a_{kt-1} \quad \cdots \quad a_{kt-2} \]

\[ a_1 \quad a_2 \quad \cdots \quad a_k \]
Claim: \( a_{kt} = kt + 1 \).

- Assume for sake of contradiction that \( a_{kt} < kt + 1 \).
Claim: \( a_{kt} = kt + 1 \).

- Assume for sake of contradiction that \( a_{kt} < kt + 1 \).
- Since \( a_{kt+1} < a_{kt} \), we have \( a_{kt+1} \neq kt + 1 \).
Outline of Proof Regarding 321 Avoidance

Claim: \( a_{kt} = kt + 1 \).

- Assume for sake of contradiction that \( a_{kt} < kt + 1 \).
- Since \( a_{kt+1} < a_{kt} \), we have \( a_{kt+1} \neq kt + 1 \).
- So, for some \( i < kt \), we have \( a_i = kt + 1 \).
Claim: \( a_{kt} = kt + 1 \).

- Assume for sake of contradiction that \( a_{kt} < kt + 1 \).
- Since \( a_{kt+1} < a_{kt} \), we have \( a_{kt+1} \neq kt + 1 \).
- So, for some \( i < kt \), we have \( a_i = kt + 1 \).
- Then, \( a_i a_{kt} a_{kt+1} \) is order-isomorphic to 321, contradiction.
Define a *consecutive block* to be a subsequence $a_i a_{i+1} \cdots a_j$ of $a_1 a_2 \cdots a_n$, such that the values $a_k$ are consecutive and in increasing order for $i < k < j$.

We remove the largest consecutive block with anchor (last value) $a_{kt}$ for each permutation in $U_{kt+1}^k(321)$; suppose that the block has length $s$. Then,
Define a *consecutive block* to be a subsequence $a_i a_{i+1} \cdots a_j$ of $a_1 a_2 \cdots a_n$, such that the values $a_k$ are consecutive and in increasing order for $i < k < j$.

We remove the largest consecutive block with anchor (last value) $a_{kt}$ for each permutation in $U_{kt+1}^k (321)$; suppose that the block has length $s$. Then,

$$a_{kt+1} a_{kt+2} a_{kt+3} a_{kt+4} \cdots$$

is sent to

$$a_{kt-t+1} a_{kt-t+2} \cdots a_{kt-s} a_{kt+1}.$$
The other direction of inserting a consecutive block is clear. Thus, the bijection holds.
The other direction of inserting a consecutive block is clear. Thus, the bijection holds.

\[
\left| U_{kt+1}^{k,1}(321) \right| = \sum_{i=k(t-1)+2}^{kt} \left| U_{i}^{k,1}(321) \right|. 
\]
Further Application of (321)-avoidance

- This gives us a nice enumeration of $U_{n}^{k,l}(321)$ for $n = kt + 1$.
- What about $n = kt + m$?
This gives us a nice enumeration of $U_{n}^{k,l}(321)$ for $n = kt + 1$.

What about $n = kt + m$?

A similar removal of a consecutive block likely holds, but the procedure of “collapsing” the highest row into the row under it may result in a row with more than $k$ elements:
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Further Application of (321)-avoidance

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- What about $n = kt + m$?

- A similar removal of a consecutive block likely holds, but the procedure of “collapsing” the highest row into the row under it may result in a row with more than $k$ elements:

\[
\begin{array}{cccc}
6 & 7 & 9 & \\
1 & 5 & 10 & 11 \\
2 & 3 & 4 & 8 \\
\end{array}
\quad 
\begin{array}{cccc}
1 & 5 & 6 & 7 & 9 \\
1 & 5 & 6 & 7 & 9 \\
2 & 3 & 4 & 8 \\
\end{array}
\]

- Thus, we will likely need to define new classes (different from $U_{n}^{k,l}$) to describe such tableaux, and so, the recursion for this case is likely more complicated, but not intractable.
Data for $l = 0$

- Now we turn to the $l = 0$ case.
Now we turn to the $l = 0$ case.

For $k = 3$:

<table>
<thead>
<tr>
<th></th>
<th>1342</th>
<th>1243</th>
<th>2341</th>
<th>3124</th>
<th>2134</th>
<th>4123</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
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<td>4</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td><strong>10</strong></td>
<td><strong>10</strong></td>
<td><strong>10</strong></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td><strong>10</strong></td>
<td><strong>10</strong></td>
<td><strong>10</strong></td>
</tr>
<tr>
<td>7</td>
<td>37</td>
<td>38</td>
<td>38</td>
<td>60</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>8</td>
<td>90</td>
<td>94</td>
<td>94</td>
<td><strong>180</strong></td>
<td><strong>190</strong></td>
<td><strong>190</strong></td>
</tr>
<tr>
<td>9</td>
<td>180</td>
<td>190</td>
<td>190</td>
<td><strong>180</strong></td>
<td><strong>190</strong></td>
<td><strong>190</strong></td>
</tr>
<tr>
<td>10</td>
<td>725</td>
<td>806</td>
<td>806</td>
<td>1330</td>
<td>1400</td>
<td>1400</td>
</tr>
</tbody>
</table>
Investigating $l = 0$

- Only avoidance patterns of a particular structure show nontrivial repetitions for $n = m$ and $n = m + 1$ for large $n$.
- Let $q$ be a permutation of length $t$ that is structurally dictated as a single down-step followed by $t - 2$ up-steps, i.e. $q = b123 \cdots (b - 1)(b + 1) \cdots (t - 1)t$ with $b \neq 1$.
- We shall call such patterns *repetitive patterns*. 
Theorem. For \( k \geq t - 1 \) and \( q \) a repetitive pattern, we have

\[
\left| U_{km+(t-2)}^{k,0}(q) \right| = \left| U_{km+(t-1)}^{k,0}(q) \right| = \left| U_{km+t}^{k,0}(q) \right| = \cdots = \left| U_{km+k}^{k,0}(q) \right|
\]

- The approach to this is a bijective proof.
- Based on the pattern \( q \), we perform an insertion of the proper value into a corresponding location.
- This serves as a surprising result for no other patterns contain repeats; for all other patterns \( q \),

\[
\left| U_{n}^{k,0}(q) \right| < \left| U_{n+1}^{k,0}(q) \right|
\]

(except for patterns of the form \( 123 \cdots t \) of course).
The result in the previous slide is quite nice, but it is very limited. However, checking numerical data indicates that a similar theorem holds for $l > 0$. 
Acknowledgements

Thanks to

- Our mentor Joel Lewis for his valuable insight and guidance.
- The PRIMES program for making this experience possible.
- Our parents for their support.

Thanks to all of you for listening.