Staged Self-Assembly

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**Introduction**

**Definition**

*Self-assembly* is the process by which order spontaneously forms from simple parts.
**Simple Parts**

**Definition**

*A tile* is a non-rotatable square with a glue on each edge.

**Definition**

Let $G$ be the set of all *glues*, including $\emptyset$, the *null glue*. 

![Diagram of a tile with glues labeled on each edge]
**Definition**

A *supertile* is a collection of tiles that are bound together. It is said to be **fully connected** if the strength of every bond is non-zero, otherwise, the supertile is **partially connected**.
**Supertiles**

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**Glues Stick Together**

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The glue function \( g : G \times G \rightarrow \mathbb{R}_0^+ \) determines the strength of the bond between glues.

- \( g(x, y) = g(y, x) \quad \forall x, y \in G \)
- \( g(\emptyset, x) = 0 \quad \forall x \in G \)
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- $g(x, x) \geq 1 \quad \forall x \neq \emptyset$
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Typically,

- $g(x, x) = 1$ $\forall x \neq \emptyset$
Tiles Stick Together

Definition

*The temperature, \( \tau \), is a property of the system that determines what strength bond is necessary to hold things together.*

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- If the strength of the bond between two tiles is at least the temperature, the tiles will connect.

- Typically we work in temperature $\tau = 1$. 
**Supertiles Stick Together**

- Two supertiles will stick together if the sum of the strengths of the bonds between all adjacent edges is at least the temperature.
- This means that (especially for $\tau = 1$) supertiles can bind together in many ways.
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SEGMENT CONSTRUCTION

- Consider the single tile:

- It will assemble into the following supertiles:
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SEGMENT CONSTRUCTION

- Consider the set of tiles:

  □ □ □ ... □ □ □
Consider the set of tiles:

It will assemble into many supertiles, including:

This supertile is said to be terminal, because it cannot bind to any other supertile.
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[ ] [ ] [ ] ... [ ] [ ]
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However, all supertiles continue connecting until they reach their final state as supertile:

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We want to minimize the number of glues because the creation of a large number of glues provides a technical challenge.

**Definition**

The *glue complexity* of a construction is $|G|$, the number of glues used, and the *tile complexity*, $T$, is the number of distinct tiles used in the construction.

$|G|/4 \leq T \leq |G|^4$, so we typically attempt to minimize the tile complexity.
SEGMENT CONSTRUCTION
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We need some way of separating groups of tiles so that not every possible connection occurs.
Bins and Stages

Definition

*A bin is a container of tiles, and a stage is a unit of time.*

- Every stage, the contents of each bin interact until they reach a terminal state.
- Then, the terminally produced supertiles from each bin can be copied and mixed into multiple other bins.
- This mixing can occur between any number of pairs of bins between each stage.
- In addition, specific tiles may be added to each bin at each stage.
The Mix Graph

Definition

Given an assembly system with $r$ stages and $b$ bins, the **mix graph** is an $rb$-vertex graph that provides a visual representation of the mixing of bins from stage to stage.
SEGMENT CONSTRUCTION ATTEMPT 3 REVISITED
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Theorem

A $1 \times n$ line segment can be constructed using 6 tiles, 7 bins, and $O(\log n)$ stages.

Can the same segment be constructed with:

- fewer bins?
- fewer tiles?
- more bins?
- more tiles?
Theorem

A $1 \times n$ line segment can be constructed using 6 tiles, 7 bins, and $O(\log n)$ stages.

Can the same segment be constructed with:

- fewer bins? yes
- fewer tiles? no
- more bins? yes
- more tiles? yes
Theorem

A $1 \times n$ line segment can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages.
$B = 2$

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B Bins

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**Theorem**

A $1 \times n$ line segment can be constructed using $T$ tiles, $B$ bins, and $O(\log_B \frac{n}{T})$ stages for $T \geq B$ and in $O(\log_T n)$ stages for $T < B$.

**Proof:**

- With more tiles, we divide them into separate groups, each with distinct glues, which allows the construction of multiple identical segments in parallel.
- This construction proceeds in $O(\log_B \frac{n}{T})$ stage complexity if there are enough tiles to create a single group of tiles.
- With less than $B$ tiles, we can make one group if we use only $T$ of the bins and leave the others empty.
Is This Optimal?

Yes it is!

Given the tiles in our final shape, an analysis of the paths they take in the mix graph gives $\Omega(\log B n T)$ stages in our construction.
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**Changing the Temperature**

**Definition**

Remember that the *temperature*, \( \tau \), is the total connection strength along the border of two supertiles that is necessary for connection to occur.

- When \( \tau = 2 \), it is useful to have some glues where \( g(x, x) = 1 \) and some where \( g(x, x) = 2 \).

**Definition**

If \( g(x, x) = 1 \), \( x \) is said to be a **single-strength** glue, while if \( g(x, x) = 2 \), \( x \) is a **double-strength** glue.

- Using \( \tau = 2 \) yields simple constructions for shapes that have more complex constructions in \( \tau = 1 \).
A $n \times n$ square can be constructed using $O(1)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$. 
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$n \times n \text{ SQUARE IN } \tau = 2$
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**Theorem**

Any monotone shape can be constructed using $O(n)$ tiles, 2 bins, and $O(\log n)$ stages in temperature $\tau = 2$.

- Construct a ‘border’ for the desired shape.
- Fill in the other parts of the shape using tiles with the same glue on all sides.
**Definition**

A shape is called **radially monotone** if, for some choice of the center, every tile can be connected to the center as a path whose lattice distance from the center is increasing.
**A Diamond Construction**

**Theorem**

*A diamond of radius $r$ can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.*
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Any radially monotone shape of radius $r$ can be constructed using $O(r)$ tiles, 1 bin, and $O(r)$ stages.
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**Arbitrary Radially Monotone Shapes**

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Why So Slow?

- Compared to our other constructions, the two constructions of arbitrary shapes have very high tile and stage complexities. Why?
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- Compared to our other constructions, the two constructions of arbitrary shapes have very high tile and stage complexities. Why?
- From an information theory perspective, an arbitrary shape encodes much more information than a segment or square, which can be described by a single number.
- In fact, using the Kolmogorov Complexity, we can show that these constructions proceed in the optimal stage complexity for their tile and bin complexities.
**Further Directions**

- Optimize construction of $n \times n$ squares for $B$ bins and $T$ tiles
- Probabilistic model
- Abnormal shapes (Extremely long rectangles, etc.)
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