On Rank Functions of Graphs

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Abstract

We study *rank functions* (also known as graph homomorphisms onto $\mathbb{Z}$), ways of imposing graded poset structures on graphs. We first look at a variation on rank functions called discrete *Lipschitz functions*. We relate the number of Lipschitz functions of a graph G to the number of rank functions of both G and $G \times \mathcal{E}$. We then find generating functions that enable us to compute the number of rank or Lipschitz functions of a given graph. We look at a subset of graphs called *squarely generated* graphs, which are graphs whose cycle space has a basis consisting only of 4-cycles. We show that the number of rank functions of such a graph is proportional to the number of 3-colorings of the same graph, thereby connecting rank functions to the Potts model of statistical mechanics. Lastly, we look at some asymptotics of rank and Lipschitz functions for various types of graphs.

1 Introduction

In this paper, we look at rankings, or rank functions. These are one of the key criteria for an adinkra, which is a graphic used in representation theory. They are interconnected with the antiferromagnetic Potts model, a model that is commonly used in particle physics. They are also, from a purely mathematical standpoint, related to partially ordered sets. In this introduction, we discuss the basics of each of these three applications, then put them together in the context of rankings.

1.1 Adinkras

An adinkra is a symbol used in West Africa to represent a concept that is not easy to define in words. Recently, the name has been coined to denote a graphical representation of supermultiplets in representation theory [6]. Mathematically, these adinkras are bipartite graphs, in which one vertex set represents *bosons* and the other represents *fermions*, which are the two types of elementary particles found in the Standard Model of physics.

Adinkras have properties beyond those of bipartite graphs. The first property is a *dashing* of the edges — a function $d : E(G) \to \{0, 1\}$ satisfying certain parity constraints [6]. The second property is a *ranking* of the vertices — a function $r : V(G) \to \mathbb{Z}$. A comprehensive study of dashings has already been done [25], and this work complements that by studying rankings.

1.2 Rankings

An *acyclic orientation* of a graph G is an assignment to every edge of G of a direction so that no cycle in G has all of its edges in the same direction. For any graph G, the number of acyclic orientations of G is equal to $(-1)^n \chi_{-1}(G)$ [19]. The number of acyclic orientations of a graph G is also equal to the number of ways in which a partially ordered set, or *poset*, structure can be imposed on G [19]. This means that for every edge $ij \in E(G)$, one of the two vertices i and j is assigned to precede the other (notated $i \geq j$). This must be done transitively; i.e. if $i \geq j$ and $j \geq k$ then $i \geq k$.

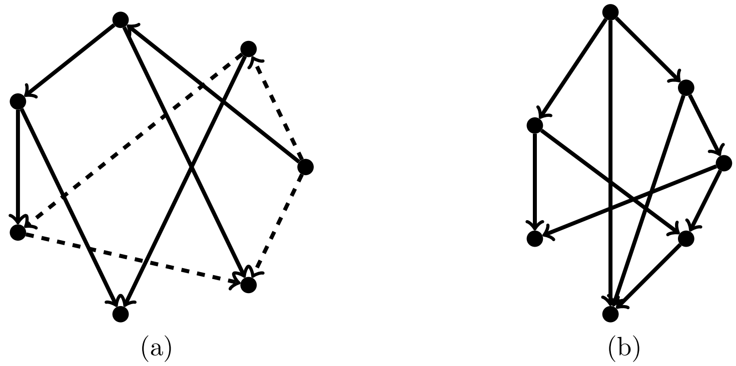

Figure 1: (a) An acyclic orientation of a graph G on 7 vertices. Each edge, or arc, is assigned a direction. There may be cycles in the graph (dashed above). However, there may not be a *directed* cycle; i.e. the edges in a cycle may not all have the same direction. (b) The poset corresponding to the orientation in (a). Note that every arrow (representing which vertex is greater) points downward.

Less well studied is the number of ways to impose a *graded* poset structure on G. This means assigning an integer *rank* $r(v)$ to each vertex so that for every edge $ij \in E(G)$, $|r(i) - r(j)| = 1$. This is equivalent to the rankings discussed in the previous section. These have primarily been studied for hypercubes [5] [9], and only a recursive algorithm was found for counting the number of rankings [25], not an explicit formula.

In this paper, we find a generating function that enables us to calculate the number of rankings of a given graph. We also look at a variation on rankings called *Lipschitz functions* and relate these to rankings. We then look at a subset of graphs called *squarely generated* graphs, which are graphs whose cycle space can be generated by 4-cycles. We show a link between rankings and 3-colorings of these same graphs, thereby also connecting rankings to the Potts model. Lastly, we look at some asymptotics for various types of graphs, which enables us to predict approximate values for the number of rankings and Lipschitz functions of many more graphs than we calculate explicitly.

1.3 Potts Model

The Potts model [1] [15] is used in statistical mechanics to characterize the behavior of systems of interacting particles. In its original incarnation [15], it is used to study collections of molecules, arranged in a two-dimensional lattice structure, by focusing on the interactions between pairs of neighboring molecules. The Potts model can be expanded to systems that do not take the shape of a lattice, and it can also be used for a wide range of situations, such as drainage of foam [10], magnetic interactions [4], entropy [11], social demographics [13] [18], and tumor growth [20] [21].

In the Potts model, each particle has a *spin* that takes on a value from 0 to $q-1$, where q depends on the system being modelled. To express the total energy (kinetic and potential) of the system, each pair of neighboring particles is assigned an *interaction energy* value. If the particles have the same spin, this value is 0; if their spins are different, the value is J, a constant specific to the system being modelled. The total energy of a given system in any state w is the sum of the energy values of every pair of particles and is expressed as $h(w) = J\sum_{i,j} 1 - \delta(\sigma_i, \sigma_j)$ [1]. Here i and j are neighboring particles, δ is the Krönecker delta, and σ_i is the spin of particle i.

There is also a *partition function* associated with the Potts model that allows us to compute the probability that the system is in a given state. The partition function is a weighted sum of the states so that those with lower energy, which are the states that are more likely to exist, have a greater weight. The partition function is $Z = \sum_w e^{-\beta h(w)}$ over all possible states w for a constant β. We can interpret β^{-1} as a sort of temperature; for example, when modelling social dynamics, the temperature can be thought of as measuring tolerance of other groups [18].

If β takes on a large enough value (i.e. at temperatures close to zero), the only states that affect the partition function's value are those with the smallest value of h. If $J > 0$, these are the states in which all particles have the same spin. This is known as a *ferromagnetic* model [1]. In this case, the partition function counts the number of states of the system in which all particles have the same spin. Since there are q possible spins, the partition function is proportional to q.

The cases that are relevant to this work are those with $J < 0$. In such cases, the states with the smallest value of h are those in which no two neighboring particles have the same spin. This is known as an *antiferromagnetic* model. In this case, the partition function counts the number of states where no two neighboring particles have the same spin. If we treat the system as a graph G, the number of such states is equal to the number of ways to color G with q colors such that no two adjacent vertices are the same color [8]. Such a coloring is called a *proper q-coloring* of G. If we hold G constant and let q vary, we find that the partition function at very low temperatures is equivalent to the chromatic polynomial.

The chromatic polynomial $\chi_q(G)$ is a well-studied graph invariant [3] [16] [23]. For a graph G on n vertices, it is an n^{th} degree polynomial in q such that for any positive integer q, $\chi_q(G)$ is the number of proper q-colorings of G. The chromatic polynomial is known to be computable in $2^n n^{O(1)}$ time, and the number of 3-colorings is known to be computable in $O\left(1.62617^n\right)$ time [7]. These 3-colorings provide the main link to the rank functions studied in this paper.

2 Preliminaries

We define a *ranking* or *rank function* as an assignment to each vertex of an integer *rank* $r(v)$ so that for every edge $ij \in E(G)$, $|r(i) - r(j)| = 1$. This is equivalent to a homomorphism r from G onto $\mathbb{Z}$. Two rankings r_1 and r_2 are considered equivalent if there is an integer η such that for every vertex v, $r_1(v) + \eta = r_2(v)$.

A (discrete) *Lipschitz function* of a graph G is an assignment to every vertex $v \in V(G)$ of an integer rank $d(v)$ such that if there is an edge $e \in E(G)$ connecting vertices v_1 and v_2, $|d(v_1) - d(v_2)| \leq 1$. This is similar to the above definition of ranking, except that two neighboring vertices may have identical rank. Equivalence between Lipschitz functions is defined as for rankings. In this paper, we will use Lipschitz functions as a tool for calculating rankings.

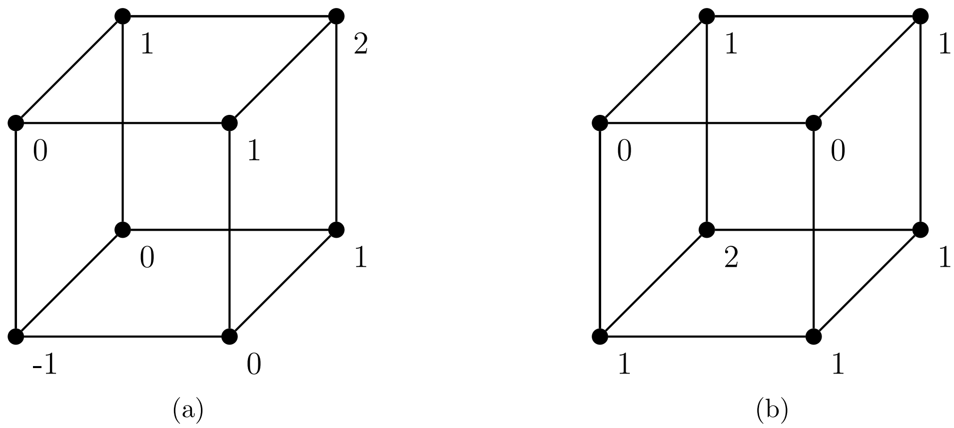

Figure 2: (a) A ranking of the hypercube I_c^3. Every pair of vertices connected by an edge has ranks that differ by exactly 1. (b) A Lipschitz function of the hypercube I_c^3. Every pair of vertices connected by an edge has either the same rank or ranks that differ by exactly 1.

For any graph G, the set of all nonequivalent rankings of G is denoted $\mathcal{R}(G)$, and the total number of rankings is denoted $|\mathcal{R}(G)|$. Similarly, the set of all nonequivalent Lipschitz functions of G is denoted $\mathcal{D}(G)$ and the total number of functions denoted $|\mathcal{D}(G)|$.

The mapping $+ : \mathcal{R}(G)\times\mathcal{R}(G) \to \mathcal{D}(G)$ is defined by $+(r_1, r_2) = d$ if $r_1(v)+r_2(v) = 2d(v)$ for all $v \in V(G)$. In section 3.2, we show that this mapping is surjective for all bipartite G and discuss the implications for rankings.

Lemma 2.0.1. *For a graph G, $\mathcal{R}(G) \neq \emptyset$ iff G is bipartite.*

Proof. First, we show that every bipartite graph has at least one ranking. Split the vertices of G into vertex sets V_0 and V_1 so that every edge connects V_0 to V_1. Define r so that $r(v) = i$ if $v \in V_i$. Every edge connects a vertex with rank 0 to a vertex with rank 1, so this is a valid ranking and $|\mathcal{R}(G)| > 0$.

Second, we show that if a graph G has at least one ranking, it must be bipartite. Partition the vertices into sets V_0 and V_1 such that V_0 (resp. V_1) contains all vertices with even (resp.

odd) rank. Since every edge connects a vertex with odd rank to a vertex with even rank, every edge connects one set to the other. So G is bipartite.

$\square$

Remark. This does not hold for Lipschitz rankings, because an edge can connect two vertices with the same rank, so that not every edge connects a vertex with odd rank to a vertex with even rank.

A *grid graph* is a graph $\mathcal{L}_{m,n}$ with mn vertices labeled $(0,0)$ through $(m-1,n-1)$ such that for every vertex (i,j), there is an edge connecting it to $(i,j+1)$ and $(i+1,j)$.

They are frequently studied by mathemeticians [12] [14] because they have a lot of symmetry, and by physicists [4] [11] [17] because they are a good approximation of 2-dimensional space. The Potts model, in fact, was originally used only on grid graphs. Some emphasis is put on grid graphs in this paper, for the same reason.

A *squarely generated graph* is a graph G whose cycle space, viewed as a vector space, has a basis consisting only of 4-cycles. A squarely generated graph must be bipartite, and therefore has a positive number of rankings. All grid graphs are squarely generated, as are all hypercubes and complete bipartite graphs. Trivially, all paths and trees are squarely generated, because their cycle space is empty.

The rankings of squarely generated graphs have a strong connection to the chromatic polynomial, as we show in Theorem 3.4.1.

3 Results

3.1 Calculations

This section consists of lemmas that will be usable later on. These lemmas pertain to calculations of simple graphs, such as trees and complete graphs. These will be used primarily

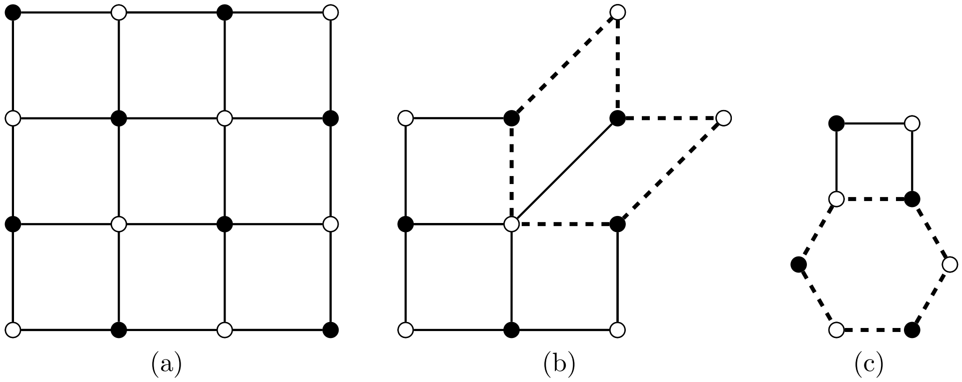

Figure 3: For all graphs, the two vertex sets are shown in white and black. (a) A 4×4 grid graph $\mathcal{L}_{4,4}$. With the exception of the vertices on the edges, every vertex has exactly four neighbors and is a part of exactly four 4-cycles. This symmetry makes it easier to calculate the number of rankings of $\mathcal{L}_{4,4}$. (b) A squarely generated graph G. Though there are cycles with 6 or more vertices (dashed), these can all be generated by 4-cycles. (c) A non-squarely generated graph G'. There is a 6-cycle (dashed) that cannot be generated by 4-cycles.

as bounds for our asymptotics (Section 3.5).

Lemma 3.1.1. *For a tree (or path) $\mathcal{T}_n$ with n vertices, we have that*

a) $|\mathcal{R}(\mathcal{T}_n)| = 2^{n-1}$

b) $|\mathcal{D}(\mathcal{T}_n)| = 3^{n-1}$.

Proof. a) We prove this by induction. For $\mathcal{T}_1$, a single vertex, there is only $2^0 = 1$ possible ranking. Therefore assume that the lemma holds true for $\mathcal{T}_n$, i.e. $|\mathcal{R}(\mathcal{T}_n)| = 2^{n-1}$. Adding one vertex to $\mathcal{T}_n$ and one edge connecting that vertex gives $\mathcal{T}_{n+1}$. **The new vertex is only attached to one vertex of $\mathcal{T}_n$, so its rank can either be one greater or one smaller than that of its attached vertex. This means that for every ranking of $\mathcal{T}_n$, there are 2 rankings of $\mathcal{T}_{n+1}$. So $|\mathcal{R}(\mathcal{T}_{n+1})| = 2|\mathcal{R}(\mathcal{T}_n)| = 2 \cdot 2^{n-1} = 2^n$, as desired.

b) The proof of part b) is similar. There is again only $3^0 = 1$ Lipschitz function for $\mathcal{T}_1$. Again assume that the lemma holds true for $\mathcal{T}_n$. Adding one vertex to $\mathcal{T}_n$ and one edge connecting that vertex gives $\mathcal{T}_{n+1}$. The new vertex is only attached to one vertex of $\mathcal{T}_n$, so its rank can be one greater, one smaller, or equal to that of its attached vertex. This means that for every ranking of $\mathcal{T}_n$, there are 3 rankings of $\mathcal{T}_{n+1}$. So $|\mathcal{R}(\mathcal{T}_{n+1})| = 3|\mathcal{R}(\mathcal{T}_n)| = 3 \cdot 3^{n-1} = 3^n$, as desired.

□

For a hypercube I_n^c, the number of rankings is unknown in general, but seems to grow exponentially with respect to the number of vertices (i.e. as k^{2^n}). Some values for small n are shown in the following table (see [25]):

n	0	1	2	3	4	5
$\|\mathcal{R}(I_n^c)\|$	1	2	6	38	990	395094

Lemma 3.1.2. *For a complete bipartite graph $K_{m,n}$, we have $|\mathcal{R}(K_{m,n})| = 2^m + 2^n - 2$.*

Proof. Let the bipartition of $V(K_{m,n})$ be V_m and V_n. Choose one vertex $v \in V_m$ and fix it at rank 0. We can split the rankings into three cases:

Case 1. $h(v') = 0$ for all $v' \in V_m$. This is the simplest case. Every vertex $u \in V_n$ can have rank either 1 or -1, independent of the others. Therefore this case contributes 2^n rankings.

Case 2. $h(v') = 2$ for at least one $v' \in V_m$. In this case, every vertex $u \in V_n$ must have rank 1. Each of the remaining $m - 1$ vertices in V_m can have either rank 0 or rank 2, as long as they do not *all* have rank 0. So this case contributes $2^{m-1} - 1$ rankings.

Case 3. $h(v') = -2$ for at least one $v' \in V_m$. This case is essentially identical to the previous case, so it also contributes $2^{m-1} - 1$ rankings.

Combining these three cases gives $|\mathcal{R}(K_{m,n})| = 2^n + 2^m - 2$, as desired. □

Lemma 3.1.3. *For a complete graph K_n, we have $|\mathcal{D}(K_n)| = 2^n - 1$.*

Proof. The proof of this lemma is almost identical to that of the previous. □

Lemma 3.1.4. *Adding an edge between two existing vertices v and v' of a graph G will always weakly reduce the number of possible rankings.*

Lemma 3.1.5. *For general G with n vertices, $2^{n/2+1} - 2 \leq |\mathcal{R}(G)| \leq 2^{n-1}$.*

Proof. A tree has minimal edges and a complete bipartite graph has maximal edges. By Lemma 3.1.4, every graph has at most as many rankings as a tree (i.e. 2^{n-1}) and at least as many as a complete bipartite (i.e. $2^{n/2+1} - 2$). $\square$

3.2 Lipschitz Functions

Here, we give results linking Lipschitz functions to rankings. We relate the number of Lipschitz functions to the number of rankings both of the same graph and of a more complex graph. These will enable us to bound the number of rankings for various families of graphs.

Theorem 3.2.1. *For any bipartite graph G, it holds that $2|\mathcal{D}(G)| = |\mathcal{R}(G \times \mathcal{E})|$.*

Proof. Let d be a Lipshitz ranking of G. We will construct exactly 2 rankings r_1, r_2 of $G \times \mathcal{E}$ that correspond to d. First select any vertex $v \in V(G)$. Let its rank be d_v. Denote the edge in $G \times \mathcal{E}$ that corresponds to v as $\mathbf{v}$. Let the vertices along $\mathbf{v}$ have ranks d_v and $d_v + 1$. Note that there are two distinct ways to do this; these will correspond to r_1 and r_2.

Choose any edge vv', and as before, let the edge in $G \times \mathcal{E}$ that corresponds to v' be $\mathbf{v}'$. The rank of v' will either be $d_v + 1$, d_v, or $d_v - 1$. Call these three cases $+$, 0, and $-$. In the first case, assign the ranks of the vertices along $\mathbf{v}'$ to be $d_v + 1$ and $d_v + 2$. In the second case, assign the ranks of those vertices to be $d_v + 1$ and d_v. In the third case, assign their ranks to be $d_v - 1$ and d_v. This is shown in the below figure.

Repeat this for other edges in G until every vertex in $G \times \mathcal{E}$ has a rank. To see that this is a valid ranking of $G \times \mathcal{E}$, choose any cycle $c \in C(G)$. Let c have $2n$ edges (since G is bipartite), of which m are labeled $+$. Since c is a cycle, m edges must also be labeled $-$, leaving $2n - 2m$ to be labeled 0. When transferring this to $G \times \mathcal{E}$, we focus on the sum of the ranks along the edge. For this to be a valid ranking, the sum at the first edge must be the same as the sum at the last edge (because they are the same edge). $+$ increases the sum by 2, $-$ decreases it by 2, and 0 keeps it constant. So going around the cycle, the final sum

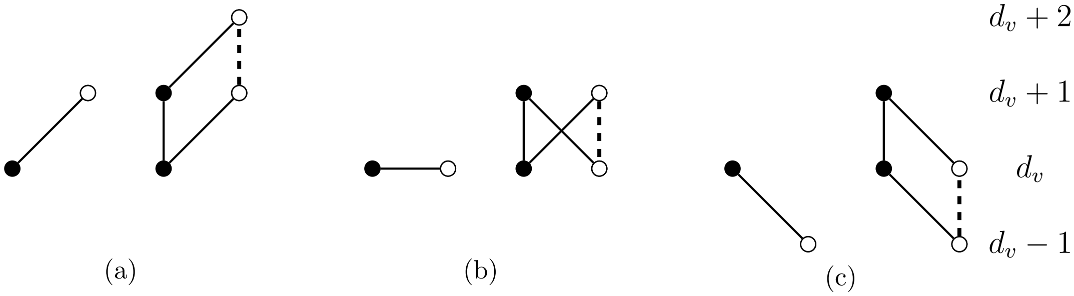

Figure 4: The three cases: (a) +, (b) 0, and (c) −. Each vertex's vertical level (labelled on the right side of the figures) is its rank. In all three figures, the 2-vertex graph represents one edge in G and the associated Lipschitz function. The 4-vertex graph represents the two edges that correspond in $G \times \mathcal{E}$ and the associated ranking. In each figure, v and $\mathbf{v}$ are in black, and v' and $\mathbf{v}'$ are white. (a) v' has higher rank than v, and both vertices of $\mathbf{v}'$ have higher rank than $\mathbf{v}$. (b) v' has the same rank as v, and the vertices of $\mathbf{v}'$ have switched rank compared to those of $\mathbf{v}$. (c) v' has lower rank than v, and both vertices of $\mathbf{v}'$ have lower rank than $\mathbf{v}$.

will be the same as the initial one. We must also have that the edges keep their orientation. Since only the 0 move flips orientation, and there must be an even number $(2n-2m)$ of such moves, the edge will keep its orientation. So this is a valid ranking of $G \times \mathcal{E}$.

□

Theorem 3.2.2. *The mapping* $+ : \mathcal{R}(G) \times \mathcal{R}(G) \to \mathcal{D}(G)$ *is surjective for all bipartite* G*.*

Proof. In the proof of the previous theorem, we constructed a ranking of $G \times \mathcal{E}$ from a Lipschitz function d on G. Split this ranking into two rankings of G. Adjust the two rankings r_1 and r_2 so that they add up to $2d$ for at least one vertex. That is, set $r_1(v) + r_2(v) = 2d(v)$ for some $v \in V(G)$. Then they will add up to $2d$ for all vertices.

To see this, recall that there are three cases: +, −, and 0. In the + case, the rankings each increase by 1 and the Lipschitz function increases by 1 as well. So we have

$$r_1(v') + r_2(v') = r_1(v) + 1 + r_2(v) + 1 = 2d(v) + 2 = 2d(v').$$

The − case is similar. For the 0 case, one of the rankings increases by 1 and one decreases by 1, while the Lipschitz function doesn't change. So in this case we have

$$r_1(v') + r_2(v') = r_1(v) + 1 + r_2(v) - 1 = 2d(v) = 2d(v').$$

This shows that the two rankings r_1 and r_2 add up to $2d$. This means that $+(r_1, r_2) = d$. So the $+$ mapping is surjective. $\square$

Corollary 3.2.3. *Since $+(r_1, r_2) = +(r_2, r_1)$, this means that there can be at most $\binom{|\mathcal{R}(G)|}{2}$ Lipschitz functions of G.*

This, combined with Theorem 3.2.1, enables us to relate $|\mathcal{R}(G)|$ and $|\mathcal{R}(G \times \mathcal{E})|$.

3.3 Generating Functions

In this section, we develop some generating functions to enable us to calculate the number of rankings or Lipschitz functions of any graph.

Theorem 3.3.1. *For every bipartite graph G, define the generating function*

$$\mathbf{R}(G) = \prod_{e \in E(G)} \left(\prod_{c \in C(G)} {y_c}^{d_e(c)} + \prod_{c \in C(G)} {y_c}^{-d_e(c)} \right),$$

where $d_E(C) = \begin{cases} 1 & \textit{if } C \textit{ and } E \textit{ go the same direction} \\ 0 & \textit{if } C \textit{ does not contain } E \\ -1 & \textit{if } C \textit{ and } E \textit{ go opposite directions} \end{cases}.$

Then the constant term of $\mathbf{R}(G)$ is $|\mathcal{R}(G)|$.

Proof. For all edges $e \in E(G)$, define

$$g(e) = \prod_{c \in C(G)} y_c^{d_e(c)} + \prod_{c \in C(G)} y_c^{-d_e(c))}.$$

The function is therefore

$$\mathbf{R}(G) = \prod_{e \in E(G)} g(e).$$

Let $n = |C(G)|$. There are 2^n terms in the expansion for $\mathbf{R}(G)$, and of these, the constant terms form a bijection with rankings of G. To see this, choose a ranking of G. Direct every edge (arc) such that the lower ranked vertex is the head of the arc. Also assign every cycle an arbitrary direction. Now choose any cycle c with $2m$ vertices, and traverse it in the cycle's direction. Of the cycle's $2m$ edges, m must go down a rank and m must go up a rank. Therefore m of the edges must be in the same direction as c ($d_e(c) = 1$) and m must be the opposite direction ($d_e(c) = -1$). Therefore in the generating function, m of the edges must contribute y_c^1 and m must contribute y_c^{-1} (and the remaining $n - 2m$ contribute y_c^0). So if this is a valid ranking, for all cycles c, the contribution to the generating function is y_c^0, and the total contribution is $y_1^0 y_2^0 \cdots y_n^0 = 1$. So every valid ranking contributes 1 to the generating function. Therefore the constant term of the generating function is exactly the number of rankings of G. $\square$

There is a similar generating function for Lipschitz functions:

Theorem 3.3.2. *For every graph G, define the generating function*

$$\mathbf{D}(G) = \prod_{e \in E(G)} \left(\prod_{c \in C(G)} {y_c}^{d_e(c)} + 1 + \prod_{c \in C(G)} {y_c}^{-d_e(c)} \right),$$

where $d_E(C) = \begin{cases} 1 & \textit{if } C \textit{ and } E \textit{ go the same direction} \\ 0 & \textit{if } C \textit{ does not contain } E \\ -1 & \textit{if } C \textit{ and } E \textit{ go opposite directions} \end{cases}.$

Then the constant term of $\mathbf{D}(G)$ *is* $|\mathcal{D}(G)|$.

The only difference between the two functions is the 1 in $\mathbf{D}(G)$. The 1 here represents the fact that an edge can have no effect on the cycles it belongs to (if the edge's vertices have the same rank).

Though these functions are nice in theory, in practice these take time to calculate, especially for very large graphs. However, this time $O\left(|V(G)|^4\right)$ is still a significant improvement on the $O\left(2^{|V(G)|}\right)$ required to calculate the number of rankings using brute force.

3.4 Squarely Generated Graphs

Squarely generated graphs are our main connection to physics and the Potts model. In this section, we create that connection by relating rankings of squarely generated graphs to colorings.

Theorem 3.4.1. *For any squarely generated graph G, $|\mathcal{R}(G)| = \frac{1}{3}\chi(G;3)$.*

Proof. Call the colors 0, 1, and 2. We demonstrate a bijection between rankings and proper colorings with one vertex fixed at color 0. First, we show that every ranking has a coloring that corresponds to it. To see this, assign every vertex v the color $r(v)$ mod 3. Since every edge connects two vertices whose ranks differ by 1, no edge will connect two vertices of the same color.

To show the reverse direction, assign a proper coloring to G. Assign the fixed vertex v_0 rank 0. Then assign rank 1 to every color 1 vertex that is connected to v_0, and rank -1 to every color 2 vertex. Continue this process outward from v_0, orienting each edge such that the vertex whose color is one greater has a rank one higher. This gives us a ranking of G, completing the proof of the bijection. □

This theorem means that the number of rankings of G is one third of the number of proper 3-colorings of $V(G)$. By symmetry, this is also equal to the number of proper 3-colorings of $V(G)$, holding one vertex fixed.

Corollary 3.4.2 ([22]). $\chi(G;n) = (-1)^{|V|+1} nT(G;1-n,0)$*, where T is the Tutte polynomial. So*

$$|\mathcal{R}(G)| = (-1)^{|V|+1}T(G;-2,0).$$

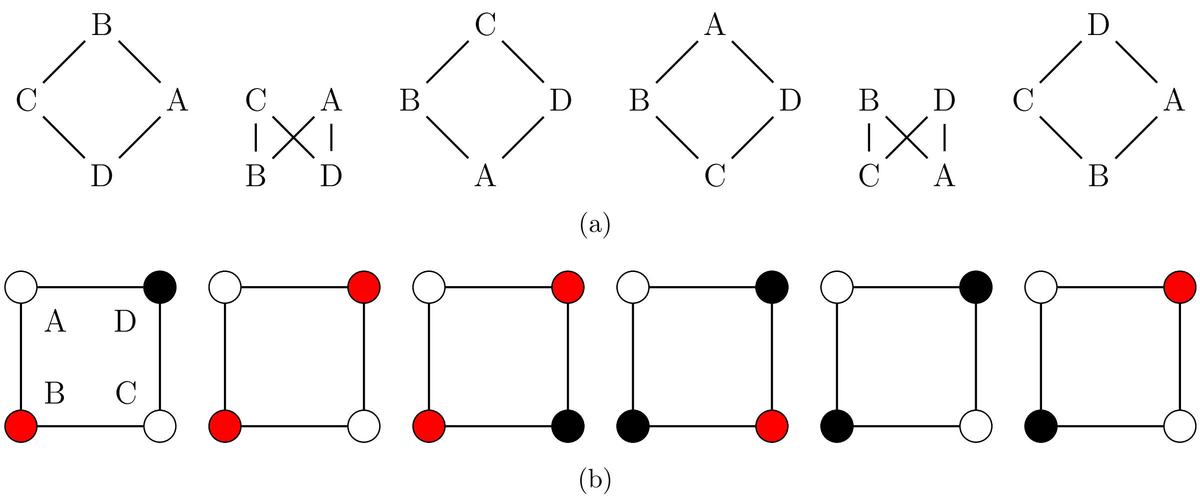

Figure 5: This is an example of the bijection between rankings and 3-colorings with one fixed vertex for the cycle $\mathcal{C}_4$. (a) These are the six rankings of the graph $\mathcal{C}_4$. (b) These are the six 3-colorings of the graph $\mathcal{C}_4$, with the upper-left corner fixed white. White = 0, red = 1, and black = 2.

Remark. For a general $m \times n$ grid graph, there is no known formula fior the number of rankings. However, for any particular grid, the number of 3-colorings (and therefore the number of rankings) can be calculated using the transfer matrix method [2].

3.5 Asymptotics

In this section, we look at asymptotics of the number of rankings of various families of graphs $\{G_n\}$, where G_n has n vertices.

Recall Lemma 3.1.5. This can be rephrased as an asymptotic:

For any G_n, $|\mathcal{R}(G_n)| = O\left(2^n\right)$ and $|\mathcal{R}(G_n)| = \Omega\left(\sqrt{2}^n\right)$.

Lemma 3.5.1 ([9], Theorem 1.4)**.** $|\mathcal{R}(I_c^d)| = O\left(2^{2^d}\right)$*, or* $|\mathcal{R}(G_n)| = O\left(2^n\right)$ *if* G_n *is a hypercube.*

By Lemma 3.1.4, any graph that is contained by I_c^d and contains $\mathcal{T}_{2^d}$ will have more rankings than I_c^d and fewer than $\mathcal{T}_{2^d}$. Both of those graphs have $O\left(2^{2^d}\right)$ rankings, giving the following corollary:

Corollary 3.5.2. *Any connected graph on 2^d vertices G_{2^d} contained by I_c^d has $O\left(2^{2^d}\right)$ rankings.*

Using the generating function, we calculate that $|\mathcal{R}(\mathcal{C}_n)| = \binom{n}{n/2}$. This is equal to $(n+1)C_{n/2}$, where C_n are the Catalan numbers. Since $C_n = O\left(\frac{4^n}{\sqrt{n^3}}\right)$, we have

Lemma 3.5.3.

$$|\mathcal{R}(\mathcal{C}_n)| = O\left(\frac{4^{n/2}}{\sqrt{n}}\right) = O\left(\frac{2^n}{\sqrt{n}}\right).$$

As a cycle has only one more edge than a path, it makes intuitive sense that its rankings would grow only slightly slower.

We also looked at asymptotics on grid graphs. [17] developed explicit formulae for $\chi_q(\mathcal{L}_{m,n})$ for all $m \leq 8$. These formulae (section 5) all have the form $\chi_q(\mathcal{L}_{m,n}) = q(q-1)V_1 M_m^{n-1} V_2^{\mathrm{T}}$, where V_1 is a row vector in q, M_m is a matrix in q, and $V_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$. Since V_1, M_m, and V_2 are independent of n, we have that $\chi_q(\mathcal{L}_{m,n}) = O\left(\lambda_{m,q}^n\right)$, where $\lambda_{m,q}$ is the largest eigenvalue of M_m for given q. This gives $|\mathcal{R}(\mathcal{L}_{m,n})| = O\left(\lambda_{m,3}^n\right)$.

For square graphs $\mathcal{L}_{n,n}$, the number of rankings grows exponentially in the order of n^2. Just as above, we have that $\chi_q(\mathcal{L}_{n,n}) = O\left(\lambda_q^n\right)$, so $|\mathcal{R}(\mathcal{L}_{n,n})| = O\left(\lambda_3^n\right)$. It was shown in [11] that $\lambda_3 = \frac{8}{9}\sqrt{3} \approx 1.5396$, so we have $|\mathcal{R}(\mathcal{L}_{n,n})| = O\left(\left(\frac{8}{9}\sqrt{3}\right)^{n^2}\right)$. We conjecture that this holds true for all rectangular grid graphs $\mathcal{L}_{m,n}$ as well:

Conjecture 3.5.4. *For any grid graph $\mathcal{L}_{m,n}$, $|\mathcal{R}(\mathcal{L}_{m,n})| = O\left(\left(\frac{8}{9}\sqrt{3}\right)^{mn}\right)$.*

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