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The Poisson algebra $(A_0, \{,\})$ retains a great deal of information about the non-commutative family $A_\hbar$. In particular, the Poisson homology $HP_0$ of $A_0$ gives an upper bound on the number of irreducible representations of the non-commutative family $A_\hbar$:

$$\#\text{Irreps}(A_\hbar) \leq \dim HP_0(A_0).$$
Michael Zhang, Yongyi Chen

*MIT PRIMES*

May 21, 2011
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- **Skew-symmetry:** $\{x, y\} = -\{y, x\}$
- **Bilinearity:** $\{z, ax + by\} = a\{z, x\} + b\{z, y\}$ for all $a, b \in \mathbb{F}$
- **Jacobi Identity:** $\{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} = 0$
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$$\{x_i, x_j \} = \{y_i, y_j \} = 0; \quad \{y_i, x_j \} = \delta_{ij};$$

where $\delta_{ij}$ is the Kronecker delta function, equal to 1 if $i = j$ and 0 otherwise.

Example:

$$\{xy, y^2 \} = x\{y, y^2 \} + y\{x, y^2 \} = 0 + y(-2y) = -2y^2.$$
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We denote by $R^G$ the invariant polynomial algebra of $R$ with respect to $G$, i.e. the set of all $r \in R$ such that $g \cdot r = r$ for all $g \in G$. 
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Let $S_2$ act on $R = \mathbb{F}[x_1, x_2, y_1, y_2]$ by permuting indices (e.g. $(12) \cdot x_1 = x_2$). Then $R^{S_2}$ is generated by the invariants $x_1 + x_2$, $y_1 + y_2$, $x_1 x_2$, $y_1 y_2$ and $x_1 y_1 + x_2 y_2$. 
Example

Let $C_n = \langle g \mid g^n = 1 \rangle$ act on $R = \mathbb{F}[x, y]$ in the following way, where $\omega$ is a primitive $n$th root of unity:

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Then $R^{C_n}$ is generated by $x^n, y^n,$ and $xy$. 
Definition

For any Poisson algebra $A$, we denote by $\{A, A\}$ the linear span of all elements $\{f, g\}$ for $f, g \in A$. 

P. Etingof and T. Schedler proved using algebraic geometric methods (D-modules) that for $F = \mathbb{C}$ or $\mathbb{Q}$, $\text{HP}_0(A)$ is finite-dimensional in many examples, including those coming from group invariants. We compute $\text{HP}_0(A)$ when $F = F_p$. In this case, $\text{HP}_0(A)$ is infinite-dimensional.
**PROBLEM STATEMENT AND PAST RESULTS**

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- We compute $HP_0$ when $\mathbb{F} = \mathbb{F}_p$. In this case, $HP_0$ is infinite-dimensional.
We form a grading

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We consider the **Hilbert Series** \( h(HP_0; t) := \sum \dim A_n t^n \).
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We consider the **Hilbert Series** \( h(HP_0; t) := \sum \dim A_n t^n \)

This is just a generating function with formal variable \( t \) formed from the grading.
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**Theorem**

If \( G = \text{Cyc}_n \) acts by
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\begin{bmatrix}
\omega & 0 \\
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where \( \omega \) is a primitive \( n \)th root of unity, for \( p > n \),
\[
h(\text{HP}_0(A); t) = \sum_{m=0}^{n-2} t^{2m} + \frac{t^{2p-2}(1 + t^{np})}{(1 - t^{2p})(1 - t^{np})}
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For small \( p \) coprime with \( n \), we prove a similar, but more complicated formula.
We have examined the 2-dimensional case $\mathbb{F}[x, y]^G$. We proved:

**Theorem**

If $G = \text{Cyc}_n$ acts by

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For small $p$ coprime with $n$, we prove a similar, but more complicated formula.
Subgroups of $SL_2(\mathbb{C})$ have integers attached called "exponents" $m_i$, and a Coxeter number $h$. 
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**Theorem**

For subgroups $G$ of $SL_2(\mathbb{C})$, and $A = \mathbb{C}[x, y]^G$, the Hilbert series of $HP_0(A)$ is: $h(HP_0; t) = \sum t^{2(m_i-1)}$
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**Conjecture**

For subgroups $G$ of $SL_2(\mathbb{C})$, and $A = \mathbb{F}_p[x,y]^G$, the Hilbert series of $HP_0(A)$ is

$$h(HP_0(A); t) = \sum t^{2(m_i-1)} + t^{2(p-1)} \frac{1 + t^h}{(1 - t^a)(1 - t^b)},$$

and $a$ and $b$ are degrees of the primary invariants.
We will try to prove the afore-mentioned conjecture for subgroups of $SL_2(\mathbb{C})$. These are the dicyclic group $Dic_n$ and the exceptional groups $E_6, E_7, E_8$. 
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We intend to extend our analysis of $HP_0$ to polynomial algebras of higher dimension, such as $\mathbb{F}[x_1, x_2, y_1, y_2]^G$. 
In MAGMA, we computed the Poisson homology of cones of smooth plane curves. Based on these computations we make the following:
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**Conjecture**

Let $A$ be the algebra $\mathbb{F}_p[x, y, z]/Q(x, y, z)$ of functions on the cone $X$ of a smooth plane curve of degree $d$ (that is, $Q$ is nonsingular, and homogeneous of degree $d$). Then,

$$h(HP_0(A); t) = \frac{(1 - t^{d-1})^3}{(1 - t)^3} + t^{p+d-3}f(t^p)$$

where

$$f(z) = (1 - z)^{-2}(2g - (2g - 1)z + \sum_{j=0}^{d-2} z^j)$$

where $g = \frac{(d-1)(d-2)}{2}$ is the genus of the curve.
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Thank you to our mentor, David Jordan, for being a great teacher, providing guidance and taking the significant time to help us out.