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In quantum physics, systems are also studied in terms of observable quantities: e.g. velocity $V$, position $P$, energy $E$, momentum $M$, ... These observables evolve through time by "Schrödinger's equations". Measurements cannot occur simultaneously, and... Heisenberg: the order of observation does matter! $PM = MP + \hbar$. Study of such systems is called "non-commutative algebra." Setting $\hbar = 0$, we recover classical physics.
Quantum spaces and non-commutative algebra

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QUANTUM SPACES AND NON-COMMUTATIVE ALGEBRA

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- There should be a special value ($\hbar = 0 \iff q = 1$) such that $A_1$ is commutative.
- We should study $A_q$ (quantum) by exporting knowledge of $A_{q=1}$ (classical), and vice versa.
A determinant formula for quantum GL(N)

Masahiro Namiki

MIT PRIMES

May 21, 2011
The determinant for $n \times n$ matrix is

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma)a^{1}_{\sigma(1)} \cdots a^{N}_{\sigma(N)}$$
**Determinants**

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\text{det}(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1)}^1 \cdots a_{\sigma(N)}^N
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Here, "sgn" is the unique homomorphism \( S_n \rightarrow \{-1, +1\} \) sending each transposition to \(-1\)
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$\text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$

$\text{Det} \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix}$ is

$$a_1^1a_2^2a_3^3 + a_2^1a_3^2a_1^3 + a_3^1a_1^2a_2^3 - a_1^1a_3^2a_2^3 - a_2^1a_1^2a_3^3 - a_3^1a_2^2a_1^3$$

Invertible matrices are characterized by non-zero determinant.
**Determinants**

The determinant for $n \times n$ matrix is

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Invertible matrices are characterized by non-zero determinant.
Definition: An algebra over \( \mathbb{C} \) is

\[ A \text{ vector space over } \mathbb{C} \]

With a multiplication map \( m: A \times A \to A \) with the properties:

\[ a \cdot (bc) = (ab) \cdot c \]
\[ a \cdot (b + c) = a \cdot b + a \cdot c \]
\[ (a + b) \cdot c = a \cdot c + b \cdot c \]
\[ a \cdot (\lambda b) = \lambda \cdot (ab) \]

With a unit 1 \( \in A \) such that

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\( e.g.) \mathbb{C} \text{ itself} \)

\( \text{Mat}_2(\mathbb{C}) \) (\( = 2 \times 2 \) matrices)

\( \mathbb{C}[x, y] \) (\( = \text{polynomials in two variables} \))

\( \mathbb{C}[x, y] / (xy = yx) \)
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  - $\mathbb{C}$ itself
  - $\text{Mat}_2(\mathbb{C})$ (= 2 × 2 matrices)
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  - $= \mathbb{C}\langle x, y \rangle/(xy = yx)$
\[ A_q(\text{Mat}_N) = \mathbb{C}\langle a^i_j \mid i = 1, 2 \cdots N, \ j = 1, 2 \cdots N \rangle / \text{Relations} \]
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The R-Matrix:
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R^{ij}_{kl} = q^{\delta_{ij}}\delta_{ik}\delta_{jl} + (q - q^{-1})\theta(i - j)\delta_{il}\delta_{jk}
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which
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\theta(s) = \begin{cases} 
1 & \text{if } s > 0 \\
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Relations: for all \( i, j = 1 \cdots N \)

\[
\sum_{k,l,m,o} R^{ij}_{kl}a^l_mR^{mk}_{no}a^o_p = \sum_{s,u,t,v} a^i_sR^{sj}_{tu}a^u_vR^{vt}_{np}
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\[ A_q(\text{Mat}_N) = \mathbb{C} \langle a^i_j \mid i = 1, 2 \cdots N, \ j = 1, 2 \cdots N \rangle / \text{Relations} \]

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Relations: for all \( i, j = 1 \cdots N \)
\[ \sum_{k,l,m,o} R_{kl}^{ij} a_m^l R_{no}^{mk} a_p^o = \sum_{s,u,t,v} a_s^i R_{tu}^{sj} a_v^u R_{np}^{vt} \]

e.g.) \[ a_1^2 a_2^1 = a_2^1 a_1^2 + (1 - q^{-2}) a_1^1 a_2^2 + (q^{-2} - 1) a_2^2 a_2^2 \]
THE QUANTUM DETERMINANT

For $q = 1$, $A_q(\text{Mat}_N) = \mathbb{C}[a^i_j | i, j = 1, \ldots N]$ is a polynomial algebra. (e.g. $a^2_1a^1_2 = a^1_2a^2_1 + (1 - q^{-2})a^1_1a^2_2 + (q^{-2} - 1)a^2_2a^2_2$)
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However it has a central element (i.e. an element which commutes with all other elements) called the quantum determinant $det_q$. 
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We sought a formula for the central element in the form:
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We sought a formula for the central element in the form:

$$z = \det_q = \sum_{\sigma \in S_n} \text{sgn}(\sigma) q^{f(\sigma)} a^1_{\sigma(1)} \cdots a^N_{\sigma(N)}.$$
Solving for $f$

for $N = 2$
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\[ \det_q = a_1^1 a_2^2 - t_{(12)} a_2^1 a_1^2 \]
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$$\Leftrightarrow (a^1_1 a^2_2 - t_{(12)} a^1_2 a^2_1) \cdot a^1_2 - a^1_2 \cdot (a^1_1 a^2_2 - t_{(12)} a^1_2 a^2_1) = 0$$
**Solving for $f$**

for $N = 2$

\[ \det_q = a_1^1 a_2^2 - t_{(12)} a_2^1 a_1^2 \]

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$$a_2^1 \cdot (a_1^1 a_2^2 - t_{(12)} a_2^1 a_1^2) - a_2^1 \cdot (a_1^1 a_2^2 - t_{(12)} a_2^1 a_1^2) + \alpha = 0$$
Solving for $f$

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In this case,

$$\alpha = (1 - q^2 + t_{(12)} - t_{(12)} q^{-2})(a_2^1 a_1^1 a_2^2 - a_2^1 a_2^2 a_2^2) = 0$$
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\alpha = (1 - q^2 + t_{(12)} - t_{(12)} q^{-2})(a_2^1 a_1^1 a_2^2 - a_2^1 a_2^2 a_2^2) = 0
\]

So, \( t_{(12)} = q^2 \), \( f((12)) = 2 \)
In order to generalize this computation for all \( n \), we need to know all the formulas for commuting two elements \( a_i^j a_m^r \).
SOLVING FOR $f$

In order to generalize this computation for all $n$, we need to know all the formulas for commuting two elements $a_i^m a_n^j$.

Using the information, we made a program which will change arbitrary order of elements in the right order.

(Right order means $(m > i) \ or \ (i = m \ and \ n > j)$).
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det_q \cdot a_2^1 - a_2^1 \cdot det_q \text{ for } N = 3, 4, 5, 6.
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**SOLVING FOR \( f \)**

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\]

We also set a program to solve the equations that we got from this.
(such as \( 1 - q^2 + t_{(12)} - t_{(12)}q^{-2} = 0 \) in \( N = 2 \))
**SOLVING FOR \( f \)**

In order to generalize this computation for all \( n \), we need to know all the formulas for commuting two elements \( a_i^ja_m^j \).

Using the information, we made a program which will change arbitrary order of elements in the right order. 

(Right order means \((m > i) \text{ or } (i = m \text{ and } n > j)\)).

We made this program organize
\[
det_q \cdot a_1^1 - a_1^2 \cdot \det_q \quad \text{for } N = 3, 4, 5, 6.
\]

We also set a program to solve the equations that we got from this.
(such as \( 1 - q^2 + t_{(12)} - t_{(12)}q^{-2} = 0 \text{ in } N = 2 \))

Thus, we got the exponents for each of the permutations.
### LIST

A part of data for $N = 4$

<table>
<thead>
<tr>
<th>Cycle notation</th>
<th>Permutation notation</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2)$</td>
<td>$[2, 1, 3, 4]$</td>
<td>$q^2$</td>
</tr>
<tr>
<td>$(2, 3)$</td>
<td>$[1, 3, 2, 4]$</td>
<td>$q^2$</td>
</tr>
<tr>
<td>$(3, 4)$</td>
<td>$[1, 2, 4, 3]$</td>
<td>$q^2$</td>
</tr>
<tr>
<td>$(1, 3, 2)$</td>
<td>$[3, 1, 2, 4]$</td>
<td>$q^3$</td>
</tr>
<tr>
<td>$(1, 3)$</td>
<td>$[3, 2, 1, 4]$</td>
<td>$q^4$</td>
</tr>
<tr>
<td>$(1, 2, 3)$</td>
<td>$[2, 3, 1, 4]$</td>
<td>$q^4$</td>
</tr>
<tr>
<td>$(1, 4, 3, 2)$</td>
<td>$[4, 1, 2, 3]$</td>
<td>$q^4$</td>
</tr>
<tr>
<td>$(1, 4, 3)$</td>
<td>$[4, 2, 1, 3]$</td>
<td>$q^5$</td>
</tr>
<tr>
<td>$(1, 3, 4, 2)$</td>
<td>$[3, 1, 4, 2]$</td>
<td>$q^5$</td>
</tr>
<tr>
<td>$(1, 2, 3, 4)$</td>
<td>$[2, 3, 4, 1]$</td>
<td>$q^6$</td>
</tr>
<tr>
<td>$(1, 2, 4)$</td>
<td>$[2, 4, 3, 1]$</td>
<td>$q^6$</td>
</tr>
<tr>
<td>$(1, 3, 4)$</td>
<td>$[3, 2, 4, 1]$</td>
<td>$q^6$</td>
</tr>
<tr>
<td>$(1, 3)(2, 4)$</td>
<td>$[3, 4, 1, 2]$</td>
<td>$q^6$</td>
</tr>
<tr>
<td>$(1, 4, 2, 3)$</td>
<td>$[4, 3, 1, 2]$</td>
<td>$q^7$</td>
</tr>
</tbody>
</table>
CONJECTURE FORMULA

By making more observations and looking at the connections between the exponents and the permutation, we predict that the formula is,

\[ \det q = \sum_{s \in S_n} (-q)^{l(s)} \cdot q^{e(s)} \cdot a_1^{s(1)} \cdots a_N^{s(N)} \]

where \( l(s) \) is the length of the permutation, which is the number of pairs out of order after \( s \).

\( i > j, s(i) < s(j) \)

\( e(s) \) is the excedance, the number of \( i \) such that \( s(i) > i \).
Conjecture Formula

By making more observations and looking at the connections between the exponents and the permutation, we predict that the formula is,

\[ \det_q = \sum_{s \in S_n} (-q)^{l(s)} \cdot q^{e(s)} \cdot a^1_{s(1)} \cdots a^N_{s(N)} \]
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$$det_q = \sum_{s \in S_n} (-q)^{l(s)} \cdot q^{e(s)} \cdot a_1^{s(1)} \cdots a_N^{s(N)}$$

$l(s)=”\text{Length of the permutation”}$
which is the number of pairs out of order after $s$.
$(i > j, s(i) < s(j))$

$e(s)=\text{excedance, the number of $i$ such that } s(i) > i.$
We confirmed our conjecture formula through $N = 11$.

We are presently working on the general proof.
First and foremost, I would like to thank David, who has really helped me throughout the program.
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Thank you all for listening to my presentation.