Fibonacci Numbers and Continued Fractions

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Continued Fractions

\[ \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ldots}}}} \]

\[ \{3; 7, 15, 1, 292, \ldots\} \]

\[ \text{Continued fractions are very useful in approximation theory.} \]

\[ \pi \approx \frac{22}{7} \]
Continued Fractions

- \( \{2; 1, 2, 1, 2, 1, \ldots\} = \{2; 1, 2\} \)
- Rational \( \iff \) Finite Continued Fraction
- Irrational \( \iff \) Infinite Continued Fraction
- Quadratic Irrational \((a + b\sqrt{c})\) \(\iff\) Periodic Infinite Continued Fraction
- Convergent: truncation of a continued fraction
Fibonacci Numbers

- $F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$

- $\frac{F_{n+1}}{F_n} = \{1, 1, 1, 1, \ldots\}$

Proof: $\frac{F_{n+2}}{F_{n+1}} = \frac{F_{n+1}}{F_{n+1}} + \frac{F_n}{F_{n+1}} = 1 + \frac{1}{\frac{F_{n+1}}{F_n}}$

- Project Goal:
  Is there a pattern for the continued fraction of $\frac{F_{n+1}}{F_n}$?
Powers of the Golden Ratio

\[
\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1 + \sqrt{5}}{2} \implies \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi^m
\]

\[
\begin{align*}
\phi^1 &= \{1; \overline{1}\} \\
\phi^2 &= \{2; \overline{1}\} \\
\phi^3 &= \{4; \overline{4}\} \\
\phi^4 &= \{6; 1, \overline{5}\} \\
\phi^5 &= \{11; \overline{11}\} \\
\phi^6 &= \{17; 1, \overline{16}\}
\end{align*}
\]

\[L_n \text{ is the } n\text{th Lucas number. } L_{n+2} = L_{n+1} + L_n, L_0 = 2, L_1 = 1\]

Theorem

\[
\begin{align*}
\phi^n &= \{L_n; \overline{L_n}\}, \text{ } n \text{ is odd} \\
\phi^n &= \{L_n - 1; \overline{1}, L_n - 2\}, \text{ } n \text{ is even}
\end{align*}
\]

\[\text{The convergents of } \phi^n \text{ are } \frac{F_{mn+n}}{F_{mn}}.\]
Squares

Examples:

- \( \frac{F_5^2}{F_4^2} = \frac{25}{9} = \{2; 1, 3, 1, 1\} \)
- \( \frac{F_6^2}{F_5^2} = \frac{64}{25} = \{2; 1, 1, 3, 1, 1, 1\} \)
- \( \frac{F_7^2}{F_6^2} = \frac{169}{64} = \{2; 1, 1, 1, 3, 1, 1, 1, 1\} \)

Theorem

\[
\frac{F_{n+1}^2}{F_n^2} = \{2; 1, 1, \ldots, \underbrace{3, 1, 1, \ldots}_{n-3 \text{ 1's}}, \underbrace{1, 1, \ldots}_{n-2 \text{ 1's}}\}
\]
Cubes

Examples:

- $\frac{F_7^3}{F_6^3} = \{4; 3, 2, 3, 2, 2, 1, 1, 1\}$
- $\frac{F_8^3}{F_7^3} = \{4; 4, 1, 1, 1, 4, 2, 2, 1, 1, 1, 1\}$
- $\frac{F_9^3}{F_8^3} = \{4; 4, 10, 4, 2, 2, 1, 1, 1, 1, 1\}$
- $\frac{F_{10}^3}{F_9^3} = \{4; 4, 3, 2, 3, 4, 2, 2, 1, 1, 1, 1, 1\}$
- $\frac{F_{11}^3}{F_{10}^3} = \{4; 4, 4, 1, 1, 1, 4, 4, 2, 2, 1, 1, 1, 1, 1\}$
- $\frac{F_{12}^3}{F_{11}^3} = \{4; 4, 4, 10, 4, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1\}$
Theorem

\[
\frac{F^n_{3n+2}}{F^n_{3n+1}} = \{4; 4, 4, \ldots, 1, 1, 1, 4, 4, \ldots, 2, 2, 1, 1, \ldots, 1\}
\]

\[
\frac{F^n_{3n+3}}{F^n_{3n+2}} = \{4; 4, 4, \ldots, 10, 4, 4, \ldots, 2, 2, 1, 1, \ldots, 1\}
\]

\[
\frac{F^n_{3n+4}}{F^n_{3n+3}} = \{4; 4, 4, \ldots, 3, 2, 3, 4, 4, \ldots, 2, 2, 1, 1, \ldots, 1\}
\]
Fourth Power (Conjecture)

\[ \frac{F_{63}^4}{F_{62}^4} = \]

\[
\begin{align*}
\{6; 1, 5, 1, \ldots\}, & \quad 14 \text{ times} \\
37, & \quad 3 \text{ times (no 33 the last time)} \\
\{3, 1, 2, 3, 2, 1, 3, 33, \ldots\}, & \quad 23 \text{ times}
\end{align*}
\]

\[ A, B, C, D, E \]

\[ \begin{align*}
\text{A consists of } 5n \text{ repetitions of 5,1 (the first is 6,1).} \\
\text{B varies with } a \mod 4: \{1\}, \{37\}, \{4, 4, 33\}, \{6, 1\}. \\
\text{C consists of } n \text{ repetitions of } 3,1,2,3,2,1,3,33 \text{ (sometimes the start or end is affected by B or D).} \\
\text{D varies with } a \mod 5: \{31, 1, 9\}, \{10, 1, 10\}, \{15\}, \{33, 3, 1, 2, 2, 1, 1\}, \{33, 4, 6\}. \\
\text{E consists of } 2\left\lfloor \frac{20n+a+2}{5} \right\rfloor \text{ repetitions of 11, if } a \equiv 2 \pmod{5}, \\
on \text{or } 2\left\lfloor \frac{20n+a+2}{5} \right\rfloor - 1 \text{ repetitions of 11, otherwise.}
\end{align*}\]

*Red varies in length, while blue varies with \(a\).*
Fifth Power (Conjecture)

\[
\begin{align*}
\frac{F^5_{17}}{F^5_{16}} &= \left\{ \frac{11; 11, 11, \text{3}, 704915, 1, 1, 5, 11, 11, \text{21}, 11, 11}{}\right\} \\
\frac{F^5_{22}}{F^5_{21}} &= \left\{ 11, 11, 11, \text{2}, 1, 86698886, 2, 5, 11, 11, 11, 21, 11, 11, 11 \right\} \\
\frac{F^5_{5n+a+1}}{F^5_{5n+a}} : \\
& A's \text{ consist of repititions of 11, whose lengths vary with } 5n. \\
& B's \text{ vary with } a, \text{ but may be compacted into one term, depending on whether } n \text{ is even or odd.} \\
& D \text{ varies with } a.
\end{align*}
\]
Fifth Power (Conjecture)

- C varies with a, but the value changes.
- For $a = 2$, the value is exceptionally large.
- $\frac{F_5^{17}}{F_5^{16}} = \{11; 11, 11, 3, 704915, 1, 1, 5, 11, 11, 21, 11, 11\}$
- $\frac{F_5^{13}}{F_5^{12}} =$
  $\{11; 11, 10, 1, 46137317, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 9, 11, 11\}$
- In addition, there is a series of 1’s rather than 11’s.
Future Research

- Fourth and fifth powers
- General theorem
- Polynomial of Fibonacci numbers
- Other Fibonacci-like sequences
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