Progress on Parallel Chip-Firing

Ziv Scully

MIT PRIMES

May 21, 2011
Motivation

- Simple rules
- “Obvious” patterns which are difficult to prove, or even wrong
- Potential connections to other fields of mathematics and science
The Parallel Chip-Firing Game

- Played on a graph
- Assign a number of chips to each vertex
- On each turn:
  - If a vertex has at least as many chips as neighbors, it *fires*
    - Otherwise, we say it *waits*
  - When a vertex fires, it gives one chip to each of its neighbors
  - Happens for all vertices in parallel
The Parallel Chip-Firing Game

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a b c d e f g h
5 1 0 0 0 2 0 0
2 2 0 1 2 0 1 0
3 0 1 1 2 1 0 0
0 1 1 2 3 1 0 0
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Basic Properties

- All games are eventually periodic
- All vertices fire the same number of times in a period
  - In a periodic-1 position, either all vertices fire or all vertices wait
- Period > 2 needs a cycle
Notation

- $\sigma(t)$ is the position after taking $t$ turns, starting with position $\sigma(0)$
- $\sigma_v(t)$ is the number of chips on vertex $v$ in position $\sigma(t)$
- $\Phi_v(t)$ is the number of $v$’s neighbors that fire at time $t$; $v$ gets one chip from each
- $F_v(t)$ is 1 if $v$ fires at time $t$ and 0 otherwise
- $c$ is the total number of chips in a position
- If $G$ is a graph, $V(G)$ is its vertex set and $E(G)$ is its edge set
Outline of Literature

- Bitar’s conjecture: maximum period $\leq$ number of vertices
- Bitar and Goles: Trees have period 1 or 2
- Kiwi et al.: Bitar’s conjecture is false!
- Dall’Asta: Period on $C_n$ divides $n$
- Levine: Period on $K_n \leq n$
- Jiang: Period on $K_{a,b} \leq 2 \min(a, b)$
Periodic or Not?

Period 4
(not periodic)

Periodic-4

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Theorem (Characterization of periodic-2 positions)

A position $\sigma(t)$ on graph $G$ is periodic-2 if and only if for all $v \in V(G)$, $\deg(v) \leq \sigma_v(t) + \Phi_v(t) \leq 2\deg(v) - 1$.

Proof.

When the period is 2, vertices alternate between firing and waiting. The above inequality is true if and only if $v$ is about to switch states.
Understanding Trees

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Understanding Trees

![Diagram of two trees with numbers at each node]

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Period 1

Period 2

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Period 1

Period 2

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Period 1

Period 2
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Theorem (Number of chips on a tree determines period)

If a game on a tree graph $G$ has $c$ chips, its eventual period is 2 if and only if $|E(G)| \leq c \leq 2|E(G)| - 1$. 
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If a game on a tree graph $G$ has $c$ chips, its eventual period is 2 if and only if $|E(G)| \leq c \leq 2|E(G)| - 1$.

Proof.

If the period is $n$, then for some time $t$, $\sigma(t)$ will be periodic-$n$.

If $n = 1$:

\[ \sigma_v(t) \leq \deg(v) - 1 \quad \text{deg}(v) \leq \sigma_v(t) \]
\[ c \leq |E(G)| - 1 \quad 2|E(G)| \leq c \]

If $n = 2$:

\[ \deg(v) \leq \sigma_v(t) + \Phi_v(t) \leq 2\deg(v) - 1 \]
\[ 2|E(G)| \leq c + \sum \frac{\Phi_v(t) + \Phi_v(t + 1)}{2} \leq 3|E(G)| - 1 \]
\[ |E(G)| \leq c \leq 2|E(G)| - 1 \]
Firing Patterns

- String of 1s and 0s indicating firing and waiting, respectively
- Classification
  - Alternating: (1, 0)
  - Sparse: not alternating, two types
    - Sparsely firing: never fires twice in a row
    - Sparsely waiting: never waits twice in a row
  - Clumpy: neither sparse nor alternating
Motors

- A special vertex with a fixed firing pattern
- Doesn’t care about receiving chips
- Natural motors
  - Subgraphs that follow normal chip firing rules
  - One key vertex behaves like a motor
    - Receiving external chips doesn’t change its firing pattern
Theorem (Periodic behavior of trees with one sparse motor)

If motor $m$ in tree graph $G$ is sparse, then for all $v \in V(G)$ at any periodic time $t$, $F_v(t) = F_m(t - d)$, where $d$ is the distance from $m$ to $v$. 

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Motorized Trees

Theorem (Periodic behavior of trees with one sparse motor)

*If motor $m$ in tree graph $G$ is sparse, then for all $v \in V(G)$ at any periodic time $t$, $F_v(t) = F_m(t - d)$, where $d$ is the distance from $m$ to $v$.***
Theorem (Periodic behavior of trees with one sparse motor)

If motor $m$ in tree graph $G$ is sparse, then for all $v \in V(G)$ at any periodic time $t$, $F_v(t) = F_m(t - d)$, where $d$ is the distance from $m$ to $v$. 

**Note:** The diagram illustrates the concept of a sparse motor in a tree graph, where the motor affects the periodic behavior of nodes in the graph.
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Constructing Natural Sparse and Alternating Motors

\((0, 1)\)
Constructing Natural Sparse and Alternating Motors

(0, 0, 5)
Constructing Natural Sparse and Alternating Motors

(0, 0, 0,

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(0, 0, 0, 0, 0, 1,
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(0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0,
Constructing Natural Sparse and Alternating Motors

(0, 0, 0, 0, 0, 1, 0, 0, 0, 0)
Constructing Natural Sparse and Alternating Motors

\[(0, 0, 0, 0, 1, 0, 0, 0, 0)\]
Further Questions

- Can a vertex have a clumpy firing pattern in a period?
- Can every vertex firing be traced back to a “driving cycle”?
- If a graph has a possible period of length $mp$ for some prime $p$, must the graph have a cycle of length $np$?
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