Hiding Behind and Hiding Inside

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A shape A can hide behind a shape B if in any direction, the shadow of B contains a translate of the corresponding shadow of A.

In 2D, all shadows are segments.
If a shape A can hide inside a shape B, then it can hide behind B.
Hiding Behind But Not Inside

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Minkowski Sum

Definition

\[ A \oplus B = \{ a + b \mid a \in A, b \in B \} \]

\[
\frac{1}{2} + \frac{1}{2} = \frac{1}{2}
\]
Theorem

For convex bodies $A$, $B$, and $C$, if $A$ and $B$ can hide behind $C$, then for any $\mu$ such that $0 \leq \mu \leq 1$, $\mu A + (1 - \mu)B$ can hide behind $C$. 
Radius of scaled disk $r = \frac{\sqrt{3}}{4} (1 - \mu)$

Side length of scaled triangle $s = \mu$

Area of Minkowski sum $= \frac{\sqrt{3}}{4} s^2 + 3rs + \pi r^2$
Best Area Ratio

\[ \mu = s = \frac{6 - \sqrt{3\pi}}{8 - \sqrt{3\pi}} \approx 0.22 \]

\[ r = \frac{3}{2(8 - \sqrt{3\pi})} \approx 0.34 \]

Ratio of areas = \[ \frac{3\pi^2 - 17\sqrt{3\pi} + 72}{(8 - \sqrt{3\pi})^2} \approx 1.39 \]
Minkowski Sum of Triangle and Inverted Triangle

\[(1 - \mu) + \mu = 1 - \mu\]
Best Area Ratio

\[
\frac{1}{2} + \frac{1}{2} = \quad \text{Area of hexagon} = \frac{3\sqrt{3}}{8}
\]

\[
\text{Area of triangle} = \frac{\sqrt{3}}{4}
\]

\[
\text{Ratio of areas} = 1.5
\]
Project Goals

- 3D shadows have shapes
- hypothesized to be impossible
- recently proved possible
- few ratio calculations

Goals:
- calculate numerical results
- improve the ratios
- prove some theorems
Minkowski Sum of Tetrahedron and Ball

\[ r = \frac{\sqrt{6} - \sqrt{2}}{4} (1 - \mu) \]
\[ s = \mu \]
\[ \alpha = \cos^{-1} \left( \frac{1}{3} \right) \]

Volume of Minkowski sum \[ = \frac{\sqrt{2}}{12} s^3 + 3(\pi - \alpha)sr^2 + \sqrt{3}s^2r + \frac{4}{3}\pi r^3 \]
Best Volume Ratio

\[ \mu \approx 0.68 \]
\[ r \approx 0.08 \]
\[ s \approx 0.68 \]

Volume of Minkowski sum \( \approx 0.13 \)

**Ratio of volumes** \( \approx 1.12 \)

The Minkowski sum hides behind the unit tetrahedron but has a bigger volume than the unit tetrahedron.
Side length of original tetrahedron $S(\triangle) = \alpha$

Side length of inverted tetrahedron $S(-\triangle) = \frac{1 - \alpha}{2}$

Volume of Minkowski sum

$$= \left[ \alpha^3 + \frac{9}{2} \alpha^2 (1 - \alpha) + \frac{9}{4} \alpha (1 - \alpha)^2 + \frac{1}{8} (1 - \alpha)^3 \right] V(\triangle)$$
α ≈ 0.77

S(△) ≈ 0.77

S(−△) ≈ 0.11

Volume of Minkowski sum ≈ 0.14

Ratio of volumes ≈ 1.16

The Minkowski sum hides behind the unit tetrahedron but has a bigger volume than the unit tetrahedron.
Suppose that $\Delta$ is an $n$-simplex and $K$ is a compact convex set in $\mathbb{R}^n$ such that the following assertions hold:

(i) Each projection $\Delta_u$ contains a translation of the corresponding projection $K_u$.

(ii) Each simplex $\Delta$ does not contain a translate of $K$.

Then there exists $t \in (0, 1)$ and a convex body $L = (1 - t)K + t\Delta$ such that the following assertions hold:

(i)$' Each projection $\Delta_u$ contains a translate of the corresponding projection $L_u$.

(ii)$' $V_n(L) > V_n(\Delta)$. 

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Hiding Behind and Hiding Inside
calculated the best volume ratio for the Minkowski sum of a tetrahedron and a ball, 1.12
found a NEW example with a better ratio — the Minkowski sum of a tetrahedron and an inverted tetrahedron, 1.16
related hiding behind and hiding inside to volume
Conjecture

The largest volume ratio for the three-dimensional case is 1.16. Furthermore, in any dimension $n$, the largest volume ratio is generated by a simplex and an inverted simplex.
Future Developments

- higher dimensions?
- simplices always generate the best ratio?
- other than Minkowski sums?
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