Hiding Behind and Hiding Inside

Christina Chen
MIT PRIMES

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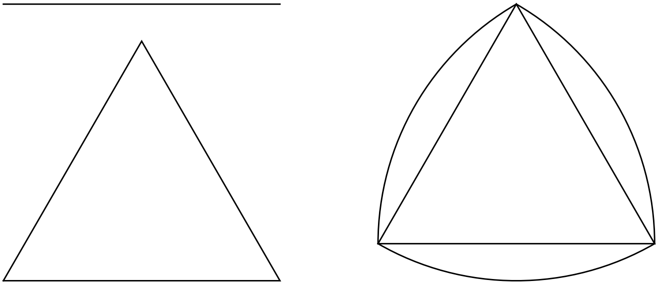
Hiding Behind

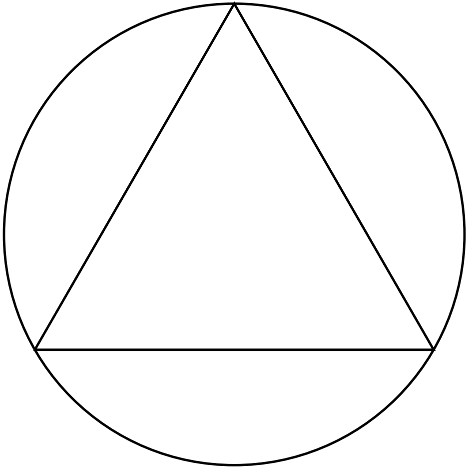
A shape A can hide behind a shape B if in any direction, the shadow of B contains a translate of the corresponding shadow of A.



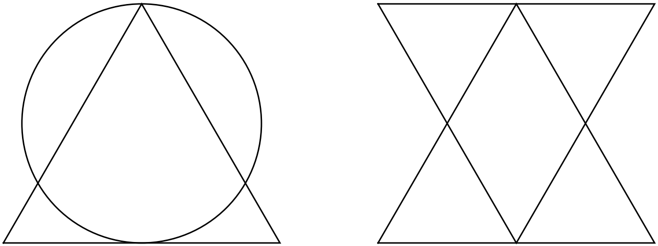
In 2D, all shadows are segments.

Hiding Inside

If a shape A can hide inside a shape B, then it can hide behind B.



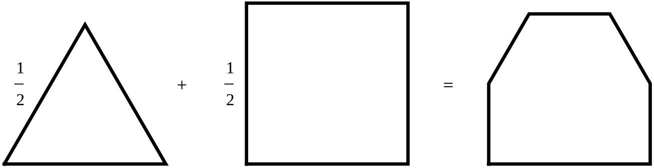
Hiding Behind But Not Inside



Minkowski Sum

Definition

$$A \oplus B = \{a + b \mid a \in A, b \in B\}$$

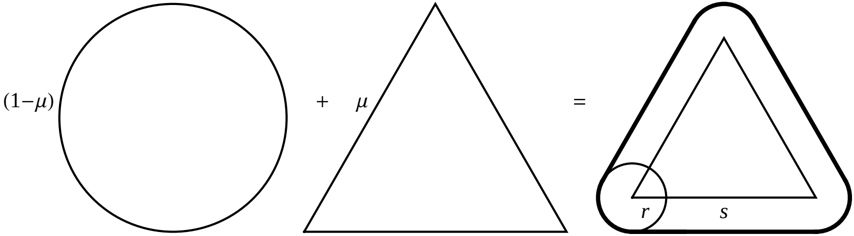


Relating Minkowski Sums To Hiding Behind

Theorem

For convex bodies A, B, and C, if A and B can hide behind C, then for any μ *such that* $0 \leq \mu \leq 1$, $\mu A + (1 - \mu)B$ *can hide behind* C.

Minkowski Sum of Triangle and Disk

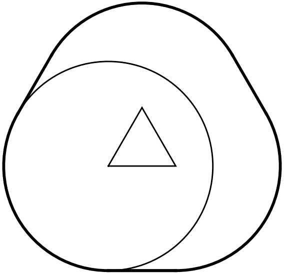


$$\text{Radius of scaled disk } r = \frac{\sqrt{3}}{4}(1-\mu)$$

$$\text{Side length of scaled triangle } s = \mu$$

$$\text{Area of Minkowski sum } = \frac{\sqrt{3}}{4}s^2 + 3rs + \pi r^2$$

Best Area Ratio

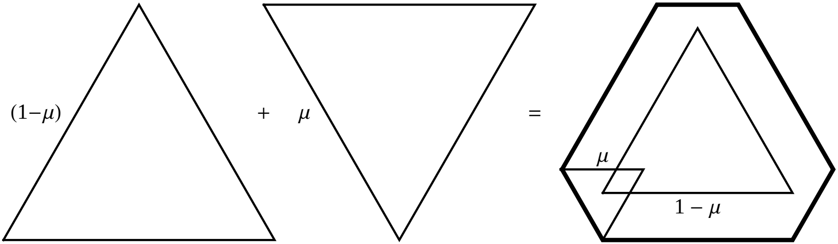


$$\mu = s = \frac{6-\sqrt{3}\pi}{8-\sqrt{3}\pi} \approx 0.22$$

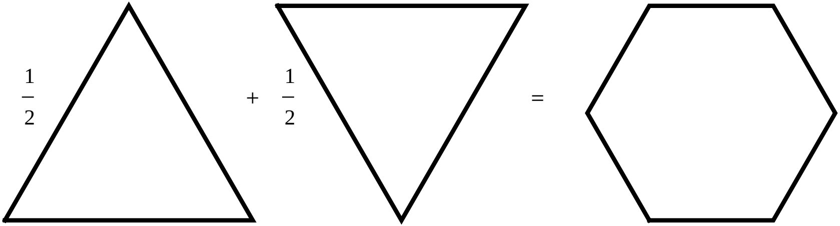
$$r = \frac{3}{2(8-\sqrt{3}\pi)} \approx 0.34$$

$$\textbf{Ratio of areas} = \frac{3\pi^2 - 17\sqrt{3}\pi + 72}{(8-\sqrt{3}\pi)^2} \approx 1.39$$

Minkowski Sum of Triangle and Inverted Triangle



Best Area Ratio



$$\text{Area of hexagon} = \frac{3\sqrt{3}}{8}$$

$$\text{Area of triangle} = \frac{\sqrt{3}}{4}$$

$$\textbf{Ratio of areas} = 1.5$$

Project Goals

- 3D shadows have shapes
- hypothesized to be impossible
- recently proved possible
- few ratio calculations

Goals:

- calculate numerical results
- improve the ratios
- prove some theorems

Minkowski Sum of Tetrahedron and Ball

$$r = \frac{\sqrt{6}-\sqrt{2}}{4}(1-\mu)$$

$$s = \mu$$

$$\alpha = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\text{Volume of Minkowski sum } = \frac{\sqrt{2}}{12}s^3 + 3(\pi-\alpha)sr^2 + \sqrt{3}s^2r + \frac{4}{3}\pi r^3$$

Best Volume Ratio

$$\mu \approx 0.68$$

$$r \approx 0.08$$

$$s \approx 0.68$$

$$\text{Volume of Minkowski sum} \approx 0.13$$

$$\textbf{Ratio of volumes} \approx 1.12$$

The Minkowski sum hides behind the unit tetrahedron but has a bigger volume than the unit tetrahedron.

Minkowski Sum of Tetrahedron and Inverted Tetrahedron

Side length of original tetrahedron $S(\triangle) = \alpha$

Side length of inverted tetrahedron $S(-\triangle) = \dfrac{1-\alpha}{2}$

Volume of Minkowski sum

$$= [\alpha^3 + \frac{9}{2}\alpha^2(1-\alpha) + \frac{9}{4}\alpha(1-\alpha)^2 + \frac{1}{8}(1-\alpha)^3]V(\triangle)$$

Best Volume Ratio

$$\alpha \approx 0.77$$

$$S(\triangle) \approx 0.77$$

$$S(-\triangle) \approx 0.11$$

$$\text{Volume of Minkowski sum } \approx 0.14$$

$$\textbf{Ratio of volumes } \approx 1.16$$

The Minkowski sum hides behind the unit tetrahdron but has a bigger volume than the unit tetrahedron.

Theorem

Theorem (jointly with T. Khovanova and D. Klain)

Suppose that Δ is an n-simplex and K is a compact convex set in $\mathbb{R}^n$ such that the following assertions hold:
(i) Each projection Δ_u contains a translation of the corresponding projection K_u.
(ii) Each simplex Δ does not contain a translate of K.
Then there exists $t \in (0,1)$ and a convex body $L = (1-t)K + t\Delta$ such that the following assertions hold:
(i)′ Each projection Δ_u contains a translate of the corresponding projection L_u.
(ii)′ $V_n(L) > V_n(\Delta)$.

Results Summary

- calculated the best volume ratio for the Minkowski sum of a tetrahedron and a ball, 1.12
- found a NEW example with a better ratio — the Minkowski sum of a tetrahedron and an inverted tetrahedron, 1.16
- related hiding behind and hiding inside to volume

Conjecture

Conjecture

The largest volume ratio for the three-dimensional case is 1.16. Furthermore, in any dimension n, the largest volume ratio is generated by a simplex and an inverted simplex.

Future Developments

- higher dimensions?
- simplices always generate the best ratio?
- other than Minkowski sums?

Acknowledgments

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