Bipartite Graphs and Matchings

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König’s Theorem

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Introduction

- Bipartite Graph: Graph whose vertices can be partitioned into two different independent sets \( U \) and \( V \) such that no edges are between any two vertices in either set.

- Perfect matching of \( U \) into \( V \): Set of edges without common vertices. Where every vertex in \( U \) is connected to an edge which goes to a distinct vertex in \( V \)
Problem

\[
\begin{array}{cccc}
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

\[a = \text{minimum \# of lines that cover all 1's}\]
\[b = \text{maximum \# of independent 1's}\]
Prove that \(a = b\).
To prove this we use a bipartite graph and translate the 1’s and 0’s to edges and the rows and columns to vertices.
We color the minimum number of lines/vertices and take all of them to the left side and all others to the right.
Now we get a new bipartite graph in which we have to find a matching from $U$ into $V$. This will represent a set of independent 1's.
Solution

We need to prove that a matching exists. For this we use Phillip Hall’s Theorem.
Philip Hall’s Theorem

Let $G$ be a bipartite graph with bipartite sets $X$ and $Y$. For a set $W$ of vertices in $X$, let $N_G(W)$ denote the neighborhood of $W$ in $G$, i.e. the set of all vertices in $Y$ adjacent to some element of $W$. The theorem in this formulation states that there is a matching that entirely covers $X$ if and only if for every subset $W$ of $X$: $|W| \leq |N_G(W)|$. 
Solution

Proof by contradiction: Suppose there exists a subset \( W \) where \( |W| > |N_G(W)| \). Then we could just switch \( W \) with \( N_G(W) \) and get a smaller set of lines! From this follows that \( b = a \)! Q.E.D.
Stable Matchings

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Requirements for stable matchings

- Bipartite Graph
- |A| = |B|
- Preferences

A

1,4,2,5,3
2,1,3,5,4
4,1,3,5,2
2,1,5,4,3
5,2,3,1,4

B

3,5,2,4,1
5,3,4,1,2
2,1,5,4,3
4,2,1,3,5
5,3,2,4,1
A perfect matching is not stable if there exist vertices $a$, $b$, $a'$, and $b'$ such that:

- There is an edge $ab$ but $a$ prefers $b'$
- There is an edge $a'b'$ but $b'$ prefers $a$

**Definition of a stable matching**

A perfect matching is not stable if there exist vertices $a$, $b$, $a'$, and $b'$ such that:

- There is an edge $ab$ but $a$ prefers $b'$
- There is an edge $a'b'$ but $b'$ prefers $a$
Gale-Shapley Algorithm

A

1,4,2,5,3
2,1,3,5,4
4,1,3,5,2
2,1,5,4,3
5,2,3,1,4

B

3,5,2,4,1
5,3,4,1,2
2,1,5,4,3
4,2,1,3,5
5,3,2,4,1
Gale-Shapley Algorithm

1. Applicants apply for favorite job.
Gale-Shapley Algorithm

1. Applicants apply for favorite job.
2. Preferred applicants are hired temporarily, rest rejected.
Gale-Shapley Algorithm

1. Applicants apply for favorite job.
2. Preferred applicants are hired temporarily, rest rejected.
3. Applicants that aren’t hired temporarily apply for next choice.
1. Applicants apply for favorite job.
2. Preferred applicants are hired temporarily, rest rejected.
3. Applicants that aren’t hired temporarily apply for next choice.
4. Repeat 2 and 3 until everyone is hired.
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3. Applicants that aren’t hired temporarily apply for next choice.

4. Repeat 2 and 3 until everyone is hired.
1. Applicants apply for favorite job.
2. Preferred applicants are hired temporarily, rest rejected.
3. Applicants that aren’t hired temporarily apply for next choice.
4. Repeat 2 and 3 until everyone is hired.
5. All applicants are hired permanently.
Proof that it produces a perfect matching

Assume the opposite is true:
• If the process is over a must have applied to b.
• From that point on b has to have a temporarily hired applicant.
• Contradiction
Proof of stability

Assume the opposite is true:
• a must have applied to b’ before applying to b.
• Either a was rejected directly or temporarily hired and then rejected. In both cases, b’ must have found a vertex x that he prefers to a.
• If x is not a’ then b’ even prefers a’ to x.
• Contradiction because it would imply that b’ prefers a’ to a.
Does the algorithm produce the best matching?

Gale-Shapley matching

1,4,3,2  2,3,4,1
2,1,4,3  3,4,1,2
3,2,1,4  4,1,2,3
4,3,2,1  1,2,3,4
Does the algorithm produce the best matching?

Gale-Shapley matching

1,4,3,2 4 1 2,3,4,1
2,1,4,3 4 1 3,4,1,2
3,2,1,4 4 1 4,1,2,3
4,3,2,1 4 1 1,2,3,4
Does the algorithm produce the best matching?

The vertex gives the edge 4 points because it’s connecting it to its favorite vertex.

Gale-Shapley matching

The vertex gives the edge 1 point because it’s connecting it to its least favorite vertex.
Does the algorithm produce the best matching?

The vertex gives the edge 4 points because it's connecting it to its favorite vertex.

Gale-Shapley matching

Points in total: 20

The vertex gives the edge 1 point because it's connecting it to its least favorite vertex.
Does the algorithm produce the best matching?

The vertex gives the edge 4 points because it's connecting it to its favorite vertex.

Gale-Shapley matching

1,4,3,2  4
2,1,4,3  4
3,2,1,4  4
4,3,2,1  4

1,4,3,2  1
2,1,4,3  1
3,2,1,4  1
4,3,2,1  1

20 Points in total

Gale-Shapley with switched roles

2,3,4,1
3,4,1,2
4,1,2,3
1,2,3,4

The vertex gives the edge 1 point because it's connecting it to its least favorite vertex.
Does the algorithm produce the best matching?

The vertex gives the edge 4 points because it’s connecting it to its favorite vertex.

Gale-Shapley matching:
- 1,4,3,2
- 2,1,4,3
- 3,2,1,4
- 4,3,2,1

Points in total: 20

Gale-Shapley with switched roles:
- 1,4,3,2
- 2,1,4,3
- 3,2,1,4
- 4,3,2,1

Points in total: 28
Does running it from both sides produce the best matching?

Gale-Shapley matching

1,2,3,4 4 1
2,3,4,1 4 1
3,4,1,2 4 1
4,1,2,3 4 1

2,4,3,1
3,1,4,2
4,2,1,3
1,3,2,4

Gale-Shapley with switched roles

1 4
1 4
1 4
20 4
Does running it from both sides produce the best matching?

Gale-Shapley matching

Gale-Shapley with switched roles

Different matching
Turan's Theorem

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Turán Graph

• graph with n-vertices.
• k sets of equal size.
• no two vertices of the same set connected.
• doesn’t contain k+1-clique.
• Turán's theorem states that the Turán graph has the largest number of edges among all $K_{k+1}$-free $n$-vertex graphs.
Assuming that:

\[ \frac{n}{k} \in \mathbb{Z} \]

\[
e(T) = \binom{k}{2} \cdot \left(\frac{n}{k}\right)^2 = \left(1 - \frac{1}{k}\right) \cdot \frac{n^2}{2}
\]
Let $G$ be maximal $K_{k+1}$-free graph, therefore it must contain $K_k$
If \( G \) has \( n \leq k \) vertices:

\[
\frac{n(n - 1)}{2} \leq \left(1 - \frac{1}{k}\right) \frac{n^2}{2}
\]

Dividing by \( n^2 \) we get:

\[
1 - \frac{1}{n} \leq 1 - \frac{1}{k}
\]
We divide the graph into two:

- $X := K_k$
- $Y := G \setminus X$

Edges:

$$\binom{k}{2} \cdot \frac{(n - k)^2}{2}$$

In $X$:

$$\left(1 - \frac{1}{k}\right) \cdot \frac{(n - k)^2}{2}$$

In $Y$:

$$(n - k)(k - 1)$$

Between $Y$ and $X$:
\[
\binom{k}{2} + (n-k)(k-1) + \left(1 - \frac{1}{k}\right) \frac{(n-k)^2}{2} = \\
= \frac{k-1}{k} \cdot \frac{n^2}{2} = \left(1 - \frac{1}{k}\right) \frac{n^2}{2}
\]
Suppose equality holds:

\[ e(G) \leq \left( 1 - \frac{1}{k} \right) \frac{n^2}{2} \]