SPANNING TREES, FORESTS AND LIMIT SHAPES

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UST on $\mathbb{Z}^2$
trunk properties (K, Wilson):

\[ \text{Prob}(\text{degree} = 2) = \frac{1}{2} \]

\[ \text{Prob}(\underbrace{\text{connecting}}_{k}) = (\sqrt{2} - 1)^k \]
Let \( d : \mathbb{R}^V \rightarrow \mathbb{R}^E \) be the incidence matrix:

\[
\begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]

Define \( \Delta = d^* Cd \), where \( C \) is a diagonal matrix of conductances.

\[
\Delta f(v) = \sum_{w \sim v} c_{vw} (f(v) - f(w))
\]

**Thm (Kirchhoff 1865)**

\[
\det \Delta_0 = \sum_{\text{sp. trees}} \prod_{e} c_e
\]

remove a row and column from \( \Delta \)
Example $G = \mathbb{Z}^2$

$$\Delta f(v) = 4f(v) - f(v + 1) - f(v + i) - f(v - 1) - f(v - i)$$

$\Delta$ is a “convolution” operator; its Fourier transform is multiplication by $P(z, w)$:

$$P(z, w) = 4 - z - \frac{1}{z} - w - \frac{1}{w}.$$ 

Q. What do the roots of $P(z, w) = \hat{\Delta}$ tell us about $\Delta$?
The Green’s function (potential kernel)

\[ G(x, y) = -\frac{1}{4\pi^2} \int_{\mathbb{T}^2} \frac{z^x w^y - 1}{4 - z - 1/z - w - 1/w} \, dz \, dw \]

For large \( x, y \), the only relevant part of \( P = 0 \) is near \((1, 1)\).
What about a slightly different setting?

Periodic conductances:

\[
\Delta = \begin{pmatrix}
5 - w - 1/w & -2 - 1/z \\
-2 - z & 5 - w - 1/w
\end{pmatrix}
\]

\[
P(z, w) = \det \Delta = w^2 + \frac{1}{w^2} - 10w - \frac{10}{w} - 2z - \frac{2}{z} + 22
\]
\( P = 0 \) has topology away from \((z, w) = (1, 1)\):

Q. What properties of spanning trees involve other points of \( P = 0 \)?

Q. Combinatorially, what is the meaning of the coefficients of \( P \)?
A. The UST is one of a two-parameter family of probability measures indexed by points \((z, w)\) on \(P(z, w) = 0\).

\[
\text{UST} \leftrightarrow (1, 1)
\]

Other points correspond to measures on “essential spanning forests”

\[
\text{sample configuration from another measure (triangular lattice)}
\]
On a strip graph,
“Cube groves” of Carroll/Speyer were discovered in the study the *cube recurrence*. (2004)
\( P(z, w) \) counts *cycle-rooted spanning forests* (CRSFs) on the torus
(subgraphs in which each component is a tree plus one edge)

**Thm:** \( P(z, w) = \sum_{\text{CRSFs } C} \left( \prod_{e} c_e \right) (2 - z^i w^j - z^{-i} w^{-j})^k, \)

where \( C \) has \( k \) cycles of homology class \((i, j)\).

e.g. \[
\begin{array}{c|c}
2 \\
\end{array}
\]

\[
P(z, w) = w^2 + \frac{1}{w^2} - 10w - \frac{10}{w} - 2z - \frac{2}{z} + 22
= 2(2 - z - \frac{1}{z}) + (2 - w - \frac{1}{w})^2 + 6(2 - w - \frac{1}{w})
\]

The set of homology classes forms a convex polygon \( N \):
the Newton polygon of \( P \), symmetric around \((0, 0)\)
If we enlarge the fundamental domain (take a cover of the torus)

\[ N_{10} : \quad \text{for the } 10 \times 10 \text{ cover } \quad (N_{10} = 10N_1) \]

Real points \((s, t)\) in \(N_1\) parametrize measures \(\mu_{s, t}\) on planar configurations with fixed average slope and density of crossings:

a random sample from \(\mu_{3,5}\).
UST

CRSFs with average slope $1/2$
**Thm:** The measures $\mu_{s,t}$ are determinantal*(for edges).

The kernel is given by

$$(K_{s,t})_{e,f} = \frac{1}{4\pi^2} \int\int_{|z|=e^x,|w|=e^y} \frac{K(z, w)_{[e],[f]} z^{x_1-x_2} w^{y_1-y_2}}{P(z, w)} \frac{dz}{iz} \frac{dw}{iw}$$

Note: The only dependence on $s, t$ is in contour of integration.

*There is a matrix $K$ such that

$$\Pr(e_1, \ldots, e_k \in T) = \det[K(e_i, e_j)_{i, j=1,\ldots,k}]$$
The free energy (growth rate) of $\mu_{s,t}$ is the growth rate of the appropriate coefficient of $P_{n \times n}(z, w)$:

$$\sigma(s, t) = \lim_{n \to \infty} \frac{1}{n^2} \log([z^{sn} w^{tn}] P_{n \times n}(z, w))$$
Legendre duality \((s, t) \leftrightarrow (x, y)\)

\[ y = \log |w| \]

\[ x = \log |z| \]

The amoeba of \(P\)

For \(s, t\) an uncritical point, \(\mu_{s,t}\) has exponential decay of correlations!
If the graph $G$ has larger fundamental domain, the phase space is richer:
Boundary connections and limit shapes

Given a region with a spanning forest connecting certain boundary vertices...
Find a *uniform* spanning forest with the same boundary connections...
More generally, start with a CRSF on a multiply connected domain

(possibly having certain boundary connections)...and find a uniform sample with same homotopy type and connections.
Limit shapes

Given a domain $U \subset \mathbb{R}^2$ and “unsigned 1-form” $|dy|$ satisfying a certain Lipschitz condition: $|dy(u)| u \in N$ for $u \in S^1$ the limit of the CRSF process $|dy_\epsilon|$ on $U \cap \epsilon \mathbb{Z}^2$ with local slope approximating $|dy|$ exists, and is the unique unsigned 1-form maximizing

$$\iint_{U} \sigma(|dy|) \, dA.$$
How to sample CRSFs with given topology?

...MCMC

add and remove “cubes”

whose faces are decorated with spanning tree edges (or dual edges).