Mumford's On the Equations Defining Abelian Varieties
Results from last talk and definitions
STAGE October 23, 2015

For the entirety of this definition sheet X will denote an abelian variety over a field k.

Fact 1. For $n \in \mathbb{Z}$, let n_X denote the map on X given by $x \mapsto nx$. Then we have

$$n_X^* \mathscr{L} \cong \mathscr{L}^{\frac{n^2+n}{2}} \otimes (-1)_X^* \mathscr{L}^{\frac{n^2-n}{2}}$$

Definition 1. Given an ample line bundle $\mathscr{L}$ on X, the *degree* of $\mathscr{L}$ is the integer d such that $H^0(X, \mathscr{L}^n) = d \cdot n^g$ for $n \geq 1$

Fact 2. Let $\pi : X \to Y$ be an isogeny of abelian varieties and $\mathscr{M}$ is an ample line bundle on Y. Set $\mathscr{L} = \pi^*(\mathscr{M})$. Then $\mathscr{L}$ is ample and $(\text{degree}\, \mathscr{M})(\text{degree}\, \pi) = (\text{degree}\, \mathscr{L})$

Definition 2. An ample line bundle $\mathscr{L}$ on X is of *separable type* if $\operatorname{char} k \nmid \text{degree}(\mathscr{L})$

Definition 3. For any line bundle $\mathscr{L}$ on X, define $H(\mathscr{L})$ to be the set of closed points x of X such that $T_x^* \mathscr{L} \cong \mathscr{L}$.

Fact 3. If $\mathscr{L}$ is an ample line bundle on X, $H(\mathscr{L})$ is finite.

Definition 4. Let $\mathscr{L}$ be an ample line bundle of separable type. Define $\mathscr{G}(\mathscr{L})$ as a set to consist of pairs (x, ϕ) where x is a closed point of X and ϕ is an isomorphism $\phi : \mathscr{L} \to T_x^* \mathscr{L}$. Then $(y, \psi) \circ (x, \phi) = (x + y, T_x^* \psi \circ \phi)$ makes $\mathscr{G}(\mathscr{L})$ a group.

Fact 4. For ample line bundles of separable type, we have a short exact sequence:

$$0 \to k^* \to \mathscr{G}(\mathscr{L}) \to H(\mathscr{L}) \to 0$$

Definition 5. For ample line bundle of separable type, $\mathscr{L}$, on X. Define $e^{\mathscr{L}} : H(\mathscr{L}) \times H(\mathscr{L}) \to k^*$ in the following way: for any x and y in $H(\mathscr{L})$ lift x and y to $\tilde{x}$ and $\tilde{y}$ respectively and set $e^{\mathscr{L}}(x, y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$.

Definition 6. Given an ample line bundle $\mathscr{L}$, a *level subgroup* of $\mathscr{G}(\mathscr{L})$ is a subgroup $\widetilde{K}$ such that the map $\mathscr{G}(\mathscr{L}) \to H(\mathscr{L})$ tkaes $\widetilde{K}$ isomorphiically onto its image or equivalently $\widetilde{K} \cap k^* = \{1\}$

Definition 7. For a level subgroup $\widetilde{K}$ whose image in $H(\mathscr{L})$ is K, define $\mathscr{G}(\mathscr{L})^*$ to be the centralizer of $\widetilde{K}$ in $\mathscr{G}(\mathscr{L})$ or equivalently to be the set of x in $\mathscr{G}(\mathscr{L})$ whose image in $H(\mathscr{L})$ lies in K

Fact 5. Let $\pi : X \to Y$ be an isogeny of abelian varieties with kernel K, and let $\mathscr{L}$ and $\mathscr{M}$ be ample line bundles on X and Y respectively such that there exists an isomorphism $\alpha : \pi^* \mathscr{M} \to \mathscr{L}$. Then the pair (π, α) determine a level subgroup $\widetilde{K}$ whose image in $H(\mathscr{L})$ is K. Futhermore, there is a canonical isomorphism $\mathscr{G}(\mathscr{M}) \cong \mathscr{G}(\mathscr{L})^* / \widetilde{K}$

Fact 6. Given, $\mathscr{L}$, an ample line bundle of separable type on X, $e^{\mathscr{L}}$ is well-defined, bilinear, and non-degenerate, and for any x in $H(\mathscr{L})$ $e^{\mathscr{L}}(x, x) = 1$.

Definition 8. Let $\mathscr{L}$ be an ample line bundle of separable type. The elementary divisors for the finite abelian group $H(\mathscr{L})$ come in pairs. Define the *type* of $\mathscr{L}$ to be $\delta = (d_1, d_2, \ldots, d_k)$ such that $d_i | d_{i+1}$ and such that $(d_1, d_1, d_2, d_2, \ldots, d_k, d_k)$ are the elementary divisors for $H(\mathscr{L})$.

Definition 9. Let $\delta = (d_1, d_2, \ldots, d_k)$ such that $d_i | d_{i+1}$. Define the following:

$$K(\delta) = \oplus_{i=1}^k \mathbb{Z}/d_i\mathbb{Z}, \quad \widehat{K(\delta)} = \operatorname{Hom}(K(\delta), k^*), \quad H(\delta) = K(\delta) \oplus \widehat{K(\delta)}$$

Furthermore define $\mathscr{G}(\delta)$ as a set to be $k^* \times K(\delta) \times \widehat{K(\delta)}$ and define a group law on $\mathscr{G}(\mathscr{L})$:

$$(\alpha, x, \ell) \cdot (\alpha', x', \ell') = (\alpha\alpha'\ell'(x), x + x', \ell + \ell')$$

Fact 7. We have the exact sequence:

$$0 \to k^* \to \mathscr{G}(\delta) \to H(\delta) \to 0.$$

Definition 10. Given a very ample line bundle of separable type, $\mathscr{L}$, a ϑ-structure on the pair $(X, \mathscr{L})$ is an isomorphism $\alpha : \mathscr{G}(\mathscr{L}) \to \mathscr{G}(\delta)$ which restricts to the identity on k^*

Definition 11. Let ι denote the map $X \to X$ given by $x \mapsto -x$. A line bundle $\mathscr{L}$ is *symmetric* if $\iota^*\mathscr{L} \cong \mathscr{L}$. As isomorphism $\phi : \iota^*\mathscr{L} \to \mathscr{L}$ is called a *normalized isomorphism* if $\phi(0)$ is the identity.

Definition 12. Let $x \in X$ be a point of order 2, $\mathscr{L}$ a symmetric line bundle, and $\phi : \iota^*\mathscr{L} \to \mathscr{L}$ a normalized isomorphism. Then define $e_*^{\mathscr{L}}$ to be the scalar α such that $\phi(x)$ is multiplication by α.

Definition 13. A line bundle $\mathscr{L}$ is *totally symmetric* if it is symmetric and for all points of order 2, $e_*^{\mathscr{L}}(x) = 1$.

Definition 14. Let $\mathscr{L}$ be a line bundle and $\psi : \mathscr{L} \to \iota^*\mathscr{L}$ be any isomorphism. For $(x, \phi) \in \mathscr{G}(\mathscr{L})$ consider the composition:

$$L \xrightarrow{\psi} \iota^*\mathscr{L} \xrightarrow{\iota^*(\phi)} \iota^*T_x^*\mathscr{L} = T_{-x}^*\iota^*\mathscr{L} \xleftarrow{T_{-x}^*} \mathscr{L},$$

and set $\delta_{-1}(x, \phi) = (-x, (T_{-x}^*\psi)^{-1} \circ (\iota^*\phi) \circ \psi)$.

For $z \in \mathscr{G}(\mathscr{L})$ and n any integer set

$$\delta_n(z) = (z)^{\frac{n^2+n}{2}} [\delta_{-1}(z)]^{\frac{n^2-n}{2}}.$$

Definition 15. For $(x, \phi) \in \mathscr{G}(\mathscr{L})$ and $n \geq 2$ let $\phi^{\otimes n}$ be the isomorphism $\mathscr{L}^{\otimes n} \to T_x^*\mathscr{L}^{\otimes n}$ induced by ϕ, and define

$$\varepsilon_n(x, \phi) = (x, \phi^{\otimes n})$$

Definition 16. Let $n \geq 2$, $\mathscr{L}$ a symmetric line bundle, and $z = (x, \phi) \in \mathscr{G}(\mathscr{L}^{\otimes n})$. Furthermore let $\psi : \mathscr{L}^{\otimes n^2} \to (n_X)^*\mathscr{L}$ be any isomorphism. Consider:

$$\begin{tikzcd}
\mathscr{L}^{\otimes n^2} \arrow[r, "\phi^{\otimes n}"] \arrow[dd, "\psi"] & T_x^*\mathscr{L} \arrow[d, "T_x^*(\psi)"] \\
 & T_x^*(n_X^*\mathscr{L}) \arrow[d, equal] \\
(n_X)^*\mathscr{L} \arrow[r, dashed] & (n_X)^*T_{nx}^*\mathscr{L}
\end{tikzcd}$$

There is a unique isomorphism $\rho : \mathscr{L} \to T_{nx}^*\mathscr{L}$ such that letting $n_X^*\rho$ be the morphism along the dotted line of the above diagram, the diagram commutes. Then set $\eta_n(x, \phi) = (nx, \rho)$.

Definition 17. Let $\delta = (d_1, \ldots, d_k)$ be such that $d_i | d_{i+1}$. We may view $K(\delta) = \oplus_{i=1}^k \mathbb{Z}/d_i\mathbb{Z}$ as a subgroup of $K(2\delta) = \oplus_{i=1}^k \mathbb{Z}/d_i\mathbb{Z}$ by the map $(a_1, \ldots, a_k) \mapsto (2a_1, \ldots, 2a_k)$. This then identifies the dual $\widehat{K(\delta)}$ as a quotient of $\widehat{K(2\delta)}$. Denote the image of ℓ under this quotient by $\overline{\ell}$. Any $\ell \in \widehat{K(\delta)}$ can be extended to a $\ell' \in \widehat{K(2\delta)}$ by $\ell'(x) = \ell(2x)$. This defines an injeciton from $\widehat{K(\delta)}$ to $\widehat{K(2\delta)}$ which we denotex $2*$, that is to say set $\ell' = 2 * \ell$

For $n \geq 2$ define: $E_2 : \mathscr{G}(\delta) \to \mathscr{G}(2\delta)$, $D_n : \mathscr{G}(\delta) \to \mathscr{G}(\delta)$, and $H_2 : \mathscr{G}(2\delta) \to \mathscr{G}(\delta)$ by:

$$E_2(\alpha, x, \ell) = (\alpha^2, x, 2 * \ell), \;\; D_n(\alpha, x, \ell) = (\alpha^{n^2}, nx, n\ell), \;\; H_2(\alpha, x, \ell) = (\alpha, 2x, \overline{\ell}).$$

Definition 18. Let $\mathscr{L}$ be an ample totally symmetric invertible sheaf'. A ϑ-structure $\alpha : \mathscr{G}(\mathscr{L}) \to \mathscr{G}(\delta)$ is called *symmetric* if $\alpha \circ \delta_{-1} = D_{-1} \circ \alpha$.

Definition 19. A pair of ϑ-structures α_1 and α_2 for $\mathscr{L}$ and $\mathscr{L}^2$ respectively said to be a *symmetric ϑ-structure for* $(\mathscr{L}, \mathscr{L}^2)$ if $\alpha_2 \circ \epsilon_2 = E_2 \circ \alpha_1$, and $\alpha_1 \circ \eta_2 = H_2 \circ \alpha_2$

Definition 20. An isomorphism $g : H(\mathscr{L}) \to H(\delta)$ is called *symplectic* if for all $z_1, z_2 \in H(\mathscr{L})$, $g(z_1) = (x_1, \ell_1)$, $g(z_2) = (x_2, \ell_2)$ and $e^{\mathscr{L}}(z_1, z_2) = \ell_2(x_1)\ell_1(x_2)^{-1}$