Lubin Tate Spectra and the Goerss Hopkins Miller Theorem

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Our goal in this section is to prove that our oriented deformation ring is a construction of E-theory. We want to reprove the theorem of Goerss-Hopkins-Miller:

Theorem (5.0.2)**.** *Let k be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbb{G}}_0$ be a finite height formal group over k. Then there is a complex periodic spectrum E and an isomorphism $\alpha : (k, \widehat{\mathbb{G}}_0) \to (\pi_0 E/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n})$ with the following properties:*

(i) The map

$$(\pi_0 E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_0}) \longrightarrow (\pi_0 E/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n}) \xrightarrow{\alpha^{-1}} (k, \widehat{\mathbb{G}}_0)$$

exhibits $(\pi_0 E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_0})$ as a universal deformation of $(k, \widehat{\mathbb{G}}_0)$. In particular, $\pi_0(E) \cong R_{LT}(\widehat{\mathbb{G}}_0)$

(ii) The ring E is $K(n)$ local and for a $K(n)$-local E_∞ ring A, the map

$$\begin{array}{c} \mathrm{CAlg}(E, A) \longrightarrow \mathcal{FG}((\pi_0 E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_0}), (\pi_0 A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_0})) \longrightarrow \mathcal{FG}((\pi_0 E/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n}), (\pi_0 A/\mathcal{J}_n^A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n})) \\ \Big\downarrow \alpha^* \\ \mathcal{FG}((k, \widehat{\mathbb{G}}_0), (\pi_0 A/\mathcal{J}_n^A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n})) \end{array}$$

is an equivalence. In particular, the mapping space $\mathrm{CAlg}(E, A)$ is discrete.

Given the existence of E theory, either property (i) or property (ii) classifies E theory. Proving that property (ii) holds will be pretty simple in our case, because it determined in a fairly straightforward way by the universal property of $R^{\mathrm{or}}_{\mathbb{G}^\circ}$. Thus, one could use the Goers-Hopkins-Miller construction and be done. However, the goal is to make an independent construction of E-theory, so we need to prove property (ii) holds too.

So far, we've constructed our candidate spectrum E:

Definition (5.1.1 and 5.1.4)**.** Let R_0 be a perfect $\mathbb{F}_p$ algebra and let $\widehat{\mathbb{G}}_0$ be a formal group over k of strict height $n < \infty$. Then let $E = E(\widehat{\mathbb{G}}_0)$ be $L_{K(n)} R^{\mathrm{or}}_{\mathbb{G}_0}$.

There is a canonical surjective map $\pi_0 R^{\mathrm{un}}_{\mathbb{G}_0} \to R_0$ with kernel $\mathcal{J}_n^{\mathbb{G}^\circ}$, so this gives a map

$$R_0 \to \pi_0 R^{\mathrm{un}}_{\mathbb{G}_0}/\mathcal{J}_n \to \pi_0 R^{\mathrm{or}}_{\mathbb{G}_0}/\mathcal{J}_n \to \pi_0 E/\mathcal{J}_n$$

The orientation on $\mathbb{G}^\circ$ over $R^{\mathrm{or}}_{\mathbb{G}_0}$ gives an orientation of $\mathbb{G}^\circ_E \simeq \widehat{\mathbb{G}}_E^{\mathcal{Q}_0}$ so we get a map α:

$$(R_0, \widehat{\mathbb{G}}_0) \to (\pi_0 E/\mathcal{J}_n^E, \mathbb{G}^\circ_{\pi_0 E/\mathcal{J}_n^E}) \to (\pi_0 E/\mathcal{J}_n^E, \widehat{\mathbb{G}}_E^{\mathcal{Q}_n})$$

In order to discuss property (i), we need α to be an isomorphism, but we can make sense of (ii) just from a map. So we'll prove (ii) first:

Theorem (5.1.5)**.** *Let R_0 be a perfect $\mathbb{F}_p$-algebra, $\widehat{\mathbb{G}}_0$ a formal group of height n over R_0 and E the Lubin-Tate spectrum. Then the map α induces an equivalence:*

$$\mathrm{CAlg}(E, A) \xrightarrow{\simeq} \mathcal{FG}((R_0, \widehat{\mathbb{G}}_0), (\pi_0 A/\mathcal{J}_n^A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n}))$$

Proof Sketch. We set up R^{un} with a universal property that says that $\mathcal{J}_n$-adic maps from R^{un} to another adic E_∞ ring A are given by $\mathbb{G}_0$-tagged p-divisible groups over A. Such a deformation is given by an ideal of definition I of A, a p-divisible group of A, a map $f\colon R_0 \to \pi_0 A/I$ and an isomorphism $f_*\mathbb{G}_0 \cong \mathbb{G}^{\mathcal{Q}}_{\pi_0 A/I}$. The space of all such $\mathbb{G}_0$-tagged p-divisible groups are a colimit over all finitely generated ideals of definition of $\pi_0 A$.

This gives a universal property for maps off of R^{or} that relates it to $\mathbb{G}_0$-tagged oriented p-divisible groups over A:

$$\mathrm{CAlg}(R^{\mathrm{or}}, A) \simeq \mathrm{colim}_I \, \mathrm{BT}_{\mathrm{or}}(A) \times_{\mathrm{BT}(\pi_0 A/I)} \mathrm{Hom}(R_0, \pi_0 A/I)$$

If $I \subseteq J$ are two ideals of definition, then J/I is nilpotent. Since R_0 is perfect, this implies $\mathrm{Hom}(R_0, \pi_0 A/I) \to \mathrm{Hom}(R_0, \pi_0 A/J)$ is a bijection, and this implies that the system is constant for ideals of definition containing the Landweber ideal $\mathcal{J}_n^A$. So this shows that $\mathrm{CAlg}(E, A)$ is equivalent to the space of maps $\mathcal{FG}((R_0, \widehat{\mathbb{G}}_0), (\pi_0 A/\mathcal{J}_n^A, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n}))$ □

Now we need to prove that condition (i) holds, which is significantly harder. Here is the statement:

Theorem (5.4.1)**.** *Let R_0 be a perfect $\mathbb{F}_p$-algebra, let $\widehat{\mathbb{G}}_0$ be a 1-d formal group of exact height n over R_0 and let E be the Lubin Tate spectrum. Then*

(a) The map α induces an isomorphism $R_0 \to \pi_0 E/\mathcal{J}_n^E$.

(b) The homotopy groups of E are in even degrees.

(c) Choose a sequence of elements $p = \overline{v}_0, \ldots, \overline{v}_{n-1}$ lifting $v_m \in \pi_* E/\mathcal{J}_m^E$. Then $\overline{v}_0, \ldots, \overline{v}_{n-1}$ is a regular sequence in $\pi_* E$.

Corollary (5.4.2)**.** *If there exists $u \in \pi_2 E$ invertible, then $\pi_* R \cong W(R_0)[\![u_1, \ldots, u_{m-1}]\!][u^{\pm}]$,*

The basic idea is that we first find the sequence $\overline{v}_0, \ldots, \overline{v}_{n-1}$ and let $E(m)$ be the E-module $\bigwedge_{i=0}^{m-1} \mathrm{cofib}(\overline{v}_i)$. We will first show that $\pi_* E(n) = R_0[u^{\pm}]$ which gives part (a) and then induct down on n to show that $E(m)$ is even for all m. This will imply (b) and (c). However, this is difficult to do with E directly because $\widehat{\mathbb{G}}_E^{\mathcal{Q}_0}$ may not have a coordinate. So instead, we apply this approach to $L_{K(n)}MP \wedge E$ over which $\widehat{\mathbb{G}}^{\mathcal{Q}_0}_{MP\wedge E}$ does have a coordinate.

So we compute the homotopy of E via diagram:

$$MP \longrightarrow L_{K(n)}MP \wedge E \longleftarrow E$$

The information we need from the map $E \to L_{K(n)}MP \wedge E$ is pretty simple – all we need to know is that it is flat and that $\mathrm{spec}\,\pi_0(E \wedge MP)$ is the scheme of coordinates on $\widehat{\mathbb{G}}_E^{\mathcal{Q}_0}$. This is because we know from the previous chapter that E is a complex periodic spectrum so the classical theory of complex cobordism and complex orientations gives this information (Jacob cites Adam's book).

On the other hand, we need a universal property for $L_{K(n)}MP \wedge E$ in the category of MP-algebras that identifies $\mathrm{CAlg}_{MP}(L_{K(n)}MP \wedge E, A)$ with some discrete space relating to maps of formal group laws.

Let $\widetilde{R}_0$ be the ring that classifies coordinates on $\widehat{\mathbb{G}}_0$, which is a faithfully flat extension of R. We produce this universal property by making a deformation of $\widetilde{R}_0$ that comes with that universal property and then checking that it is equivalent to $L_{K(n)}MP \wedge E$.

Since $\widetilde{R}_0$ has a coordinate, there is a map $L \to \widetilde{R}_0$ classifying $\widehat{\mathbb{G}}_0$ as a formal group law. Because $\widetilde{R}_0$ is height n, this factors through L/I_n.

Definition (5.2.1)**.** Let A be a connective E_∞ ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0 A/I$. Suppose we are given a diagram of connective E_∞ rings

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

where B_0 is a discrete ring. Then f is an A-thickening of f_0 if the following conditions hold:

(a) The E_∞ ring B is I-complete.

(b) The diagram induces an isomorphism $\pi_0 B/I\pi_0 B \to B_0$.

(c) If R is a connective E_∞ algebra over A which is I-complete, the canonical map

$$\mathrm{CAlg}_A(B, R) \to \mathrm{CAlg}^{\heartsuit}_{A_0}(B_0, \pi_0 R/I\pi_0 R)$$

is a homotopy equivalence. In particular, the mapping space $\mathrm{CAlg}_A(B, R)$ is discrete.

We need a result that tells us when such a thickening exists.

Theorem (5.2.5)**.** *Let A be a connective E_∞ ring and $I \subseteq \pi_0 A$ be a finitely generated ideal. Let $A_0 = \pi_0 A/I$. Suppose that A_0 is an $\mathbb{F}_p$-algebra which is almost perfect as an A-module and the Frobenius map $\phi_{A_0} : A_0 \to A_0$ is flat. Let $f : A_0 \to B_0$ be a relatively perfect map of commutative A-modules. Then there exists a diagram:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

which exibits f as an A-thickening of f_0 and is a pushout square.

Theorem 5.2.5 is proven by deformation-theoretic techniques similar to the ones in Chapter 3 that Jeremy and Allen talked about.

Lemma (5.4.6)**.** *There is a diagram:*

$$\begin{array}{ccc} \tau_{\geq 0}MP & \longrightarrow & \widetilde{A} \\ \downarrow{\scriptstyle \rho} & & \downarrow \\ L/I_n & \xrightarrow{\rho_0} & \widetilde{R}_0 \end{array}$$

exhibiting ρ as an $\tau_{\geq 0}MP$-thickening of ρ_0.

To check this, we need to show that L/I_n is a perfect module over $\tau_{\geq 0}MP$, that ρ_0 is relatively perfect, and that L/I_n has a flat Frobenius map. The first condition comes down to the fact that $L = \pi_0 MP$ is the cofiber of $u\colon \Sigma^2\tau_{\geq 0}MP \to \tau_{\geq 0}MP$ and $v_0, \ldots v_n$ are a regular sequence generating I_n. The remaining two conditions are purely commutative algebra. The statement that ρ_0 is relatively perfect follows from proposition 4.4.23 that the extension of scalars function $\mathrm{FGroup}^{=n}(R) \to \mathrm{FGroup}^{=n}(R^{1/p^{\infty}})$ is an equivalence of categories.

We define $A = \widetilde{A}[u^{-1}]$, which is an MP-algebra. Recall that Sanath proved:

Theorem (4.5.2)**.** *Let A be a p-local complex periodic E_∞-ring and n a positive integer. Then A is $K(n)$-local if and only if:*

(a) A is complete with respect to $\mathcal{J}_n^A \subseteq \pi_0 A$

(b) The $(n+1)$st Landweber ideal $\mathcal{J}_{n+1}^A$ is the unit ideal. Equivalently, $\widehat{\mathbb{G}}_0$ has height at most n.

Lemma. *The E_∞ ring A is even periodic and $K(n)$-local.*

Proof. Considering the cofiber sequence $u\colon \Sigma^2\widetilde{A} \to \widetilde{A} \to \pi_0\widetilde{A}$ we see that $\pi_*\widetilde{A} = \pi_0\widetilde{A}[u]$ and that $\pi_* A = \pi_0\widetilde{A}[u^{\pm}]$. To check $K(n)$-locality, we use theorem 4.5.2. By construction A is I_n-complete, and because $\widehat{\mathbb{G}}_0$ is of exact height n, it has height at most n, and so I_{n+1} is the unit ideal. □

There is a canonical map $\gamma_0\colon (R_0, \widehat{\mathbb{G}}_0) \to (\widetilde{R}_0, \widehat{\mathbb{G}}_A^{\mathcal{Q}_n})$ so by theorem 5.1.5 there is a lift to a map $\gamma\colon E \to A$.

Proposition (5.4.10)**.** *The maps $\rho\colon MP \to A$ and $\gamma\colon E \to A$ exhibit A as $L_{K(n)}MP \wedge E$.*

Proof. We want to show that the map $\theta\colon \mathrm{CAlg}_{MP}(A, B) \to \mathrm{CAlg}(E, B)$ is an equivalence. Let $B_0 = \pi_0 B/I_n\pi_0 B$. The map $MP \to B$ determins a coordinate t on $\widehat{\mathbb{G}}_B^{\mathcal{Q}_n}$. By theorem 5.15, $\mathrm{CAlg}(E, B) \cong \mathcal{FG}((R_0, \widehat{\mathbb{G}}_0), (B_0, \widehat{\mathbb{G}}_B^{\mathcal{Q}_n}))$. The restriction map $\mathrm{CAlg}_{MP}(A, B) \to \mathrm{CAlg}_{\tau_{\geq 0}MP}(\widetilde{A}, \tau_{\geq 0}B)$ is a homotopy equivalence, so we can use the definition of ρ as a thickening of $\rho_0\colon L/I_n \to \widetilde{R}_0$ to get $\mathrm{CAlg}_{MP}(A, B) \simeq \mathrm{CAlg}^{\heartsuit}_{L/I_n}(\widetilde{R}_0, B_0)$. So θ corresponds to θ' in the commutative diagram of sets:

$$\begin{array}{ccc} \mathrm{CAlg}^{\heartsuit}_{L/I_n}(\widetilde{R}_0, B_0) & \xrightarrow{\quad\theta'\quad} & \mathcal{FG}((R_0, \widehat{\mathbb{G}}_0), (B_0, \widehat{\mathbb{G}}_B^{\mathcal{Q}_n})) \\ & \searrow \quad \swarrow & \\ & \mathrm{CAlg}^{\heartsuit}(R_0, B_0) & \end{array}$$

To show that θ' is bijective, it suffices to take the fiber over a particular homomorphism $g : R_0 \to B_0$. Then this is saying that coordinates on $\widehat{\mathbb{G}}_B^{\mathcal{Q}_n}$ are equivalent to automorphisms of $\widehat{\mathbb{G}}_B^{\mathcal{Q}_n}$. The map sends a coordinate s to the unique automorphism that takes our standard coordinate t to s. □

Proof of theorem 5.4.1. Let $E(m) = \bigwedge_{i<m} \mathrm{cofib}(\overline{v}_m)$ and let $A(m) = E(m) \wedge_E A$. In $\pi_* A$ we can form $u^{1-p^m}\overline{v}_m \in \pi_0 A$. By a standard regular sequences argument, $A \to A(m)$ induces an isomorphism $\pi_* A/I_m\pi_* A \cong \pi_* A(m)$. The quotient $\pi_* A/I_n\pi_* A$ we have already identified as $\widetilde{R}_0[u^{\pm}]$ so this gives an isomorphism of $\pi_* A$-modules $\pi_* A(n) \cong \widetilde{R}_0[u^{\pm}]$. We have that $A(m) \simeq L_{K(m)}(MP \wedge E(m))$. The smash product $MP \wedge E(n)$ is already $K(n)$-local because it is $\mathcal{J}_n^E$ complete so $\pi_*(MP \wedge E(n)) \cong \widetilde{R}_0[u^{\pm}]$. We see that

$$MP \wedge E(n) \simeq (MP \wedge E) \wedge_E E(n)$$

has even homotopy groups so by the flatness of $MP \wedge E$ over E, so does $E(n)$. By inducting downwards and chasing regular sequences, we deduce that $E(m)$ has even homotopy groups for each m. From this, we deduce part (b) that $E(0)$ is even and part (c) that the sequence $\overline{v}_0, \ldots, \overline{v}_{n-1}$ is regular in $\pi_* E$. To prove (a), we need to show that $\psi\colon R_0 \to \pi_0 E/\mathcal{J}_n^E$ is an isomorphism. Recall that the map $R_0 \to \widetilde{R}_0$ is faithfully flat, so we can check this after tensoring up to $\widetilde{R}_0$. This gives

$$\widetilde{R}_0 \to \widetilde{R}_0 \otimes_{R_0} (\pi_0 E/\mathcal{J}_n^E) \cong \widetilde{R}_0 \otimes_{R_0} \pi_0 E(n) \cong \pi_0(MP \wedge E(n))$$

which we've already checked is an isomorphism. □