VERY ROUGH NOTES ON SPECTRAL DEFORMATION THEORY

1. A bit of review

To the best of my understanding, let's understand what we discussed so far:

Suppose R is a putative $\mathbb{E}_\infty$-ring we would like to build. There are various ways to try construct R. We talked a lot about them in a previous semester of juvitop.

(1) One can try to present R as the Thom spectrum of an infinite loop map.
(2) One can filter the $\mathbb{E}_\infty$-operad. This leads to Robinson or Goerss–Hopkins obstruction theory, with Goerss–Hopkins obstruction theory involving the added wrinkle of simultaneously building up the underlying spectrum. This is the classically easiest way to build Morava E-theory.
(3) One can attach $\mathbb{E}_\infty$-cells, essentially presenting the ring by $\mathbb{E}_\infty$-generators and relations. This can yield some beautiful filtrations related to what we study in the Thursday seminar, such as the Arone–Lesh–Rognes filtration

$$F_0 = (Q\mathbb{CP}^\infty)^\gamma \longrightarrow F_1 \longrightarrow \cdots \longrightarrow MU.$$

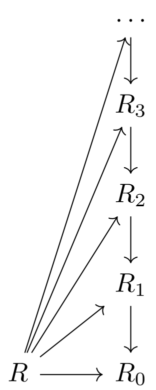
Sanders gave a talk in the Monday seminar about how such cell structures provide homological stability results.
(4) A final way is to build the ring by a sequence of square-zero extensions. This works best when building connective $\mathbb{E}_\infty$-ring spectra R, which we'll focus on now. The classical Postnikov tower of R is a sequence of square-zero extensions, for instance, which Basterra and Mandell use to prove that BP is an $\mathbb{E}_4$-ring spectrum.

Jacob's new approach is to make a rather different looking sequence of square-zero extensions converging to a connective adic $\mathbb{E}_\infty$-ring R.

Starting with a ring R_0 under R, we may try to build a new ring via a square-zero extension that is *the universal square-zero extension* a little bit closer to R. To do this, we use the canonical map $L_{R_0} \longrightarrow L_{R_0|R}$.

This makes a square-zero extension R_1, and then we can repeat the process, forming R_2 from $L_{R_1} \longrightarrow L_{R_1|R}$, etc.

We produce a tower of square-zero extensions



Crucially, to build this tower we do not need to have the ring R in hand. We just need a recipe for transforming rings R_i into R_i-module spectra $L_{R_i|R}$ together with maps

$$L_{R_i} \longrightarrow L_{R_i|R}.$$

It then becomes necessary to understand convergence of the tower. How might we understand its limit, or if we have an $\mathbb{E}_\infty$-ring R already in hand, how do we check that the natural map

$$R \longrightarrow \lim R_i$$

is an equivalence?

Jacob's answer is to look at the comparison map

$$\mathrm{Hom}(\lim R_i, S) \longrightarrow \mathrm{Hom}(R, S)$$

for various $\mathbb{E}_\infty$-rings S. For truncated rings S (i.e. rings with homotopy groups in a small range of degrees), we can check that these two mapping spaces are the same. Roughly speaking, that's because through the eyes of S the tower becomes constant.

Let's see an example. Suppose we start with $R_0 = H\mathbb{F}_p$ and $R = \mathbb{S}_p$, the p-complete sphere spectrum. Jacob spends a lot of pages on this example because $\mathbb{S}_p$ is the universal unoriented deformation of $\mathbb{G}_m$ over $\mathbb{F}_p$. If S is any $\mathbb{F}_p$-algebra with homotopy groups concentrated in degree 0, then any map $R_i \longrightarrow S$ will factor through $H\mathbb{F}_p \simeq R_0$. So we know that the comparison map is an equivalence at least for this class of S.

One can slowly build up the class of rings S on which one knows the functor of points to be an equivalence. For suitably truncated S, with finitely many finite homotopy groups, I believe that the tower eventually looks constant.

To finish these convergence proofs, and understand what happens for non-truncated S, Allen explained that it was essential that each $L_{R_i|R}$ be a suitably finite thing. In technical terms, we want $L_{R_i|R}$ to be an almost perfect R_i-module. For the sorts of functors $- \mapsto L_{-,R}$ that show up in Jacob's paper, this just comes down to checking that L_{R_0} be almost perfect.

2. The F-finiteness theorem

Suppose that A is a *coherent*, connective $\mathbb{E}_\infty$-ring. The word coherent just means that

1. Every finitely generated ideal in $\pi_0(A)$ is finitely presented.
2. The group $\pi_n(A)$ is a finitely presented $\pi_0(A)$-module for each $n > 0$.

Noetherian rings are coherent, but $\mathbb{Z}[x_1, x_2, \cdots]$ is coherent without being Noetherian.

Remark 2.1. If A is a classical coherent ring, then finitely generated submodules of finitely presented modules are finitely presented. If M and N are finitely presented A-modules, then the kernel and cokernel of any map $f : M \to N$ is also finitely presented. One proof of this sort of thing proceeds by first checking the case when either M or N is free, and then diagram chasing elements along exact sequences that express finite presentation, like in the proof of the five-lemma.

Definition 2.2. The category of almost perfect A-modules is the smallest subcategory of $\mathbf{LMod}_A$, closed under retracts, shifts, and finite colimits, that contains any geometric realization $|P_\bullet|$ of finite rank free R-modules.

Proposition 2.3. *An A-module M is almost perfect if and only if*

1. $\pi_m M = 0$ *for all sufficiently small* m.
2. *For every integer m, $\pi_m M$ is a finitely presented $\pi_0 A$-module.*

Proof. A fairly simple proof can be found in Higher Algebra. Let's at least remark here why it is reasonable that almost perfect modules satisfy both properties. It is obvious that the class of modules satisfying both properties is closed under retracts and shifts. The homotopy groups of finite colimits and geometric realizations can be analyzed by spectral sequences, using the previous remarks and the fact that finitely presented modules are closed under extension. □

Corollary 2.3.1. *$\mathbb{Z}$ is an almost perfect $\mathbb{S}$-module.*

Remark 2.4. The ring $\mathbb{S}$ is obviously coherent, in the above sense, but the ordinary ring $\pi_*(\mathbb{S})$ is far from Noetherian. Note that if it were Noetherian then the ideal of positive degree classes would be finitely generated, and in fact these generators would have to generate $\pi_*(\mathbb{S})$ as a $\mathbb{Z}$-algebra. Since every positive-degree element of the homotopy groups of spheres is nilpotent, this is impossible.

The main thing for today will be to prove the following theorem, whose proof is spread over 2 sections of Chapter 3 of Lurie's :

Theorem 2.5. *Let R denote a classical Noetherian $\mathbb{F}_p$-algebra. Then the following conditions are equivalent:*

(1) *The absolute cotangent complex L_R is an almost perfect R-module.*
(2) *The algebraic cotagent complex L_R^{alg} is an almost perfect R-module.*
(3) *The algebra R is F-finite, meaning the Frobenius $\phi : R \longrightarrow R$ presents R as a finitely generated R-module.*

We'll also try to better understand what kinds of rings are F-finite, and in particular why F-finiteness is preserved as we go up a tower of nice square-zero extensions.

First, let's talk about (i) vs (ii). Once we prove the equivalence of (i) and (ii) we'll be in great shape, because (ii) and (iii) are both purely algebraic conditions and we might consider algebra a bit simpler than homotopy theory.

How can we show that (i) and (ii) are equivalent? It is a formal result in SAG that, for any morphism $A \to B$ of classical commutative rings,

$$L_{B/A}^{alg} \simeq B \otimes_{B\otimes_{\mathbb{S}}\mathbb{Z}} L_{B/A}.$$

Using this, the result follows from the fact that $\mathbb{Z}$ is an almost perfect $\mathbb{S}$-module.

Now, why are (ii) and (iii) related? Well, the more important direction for our purposes is that (iii) implies (ii), so I'll do that direction.

The fiber sequence $R \otimes_{\mathbb{F}_p} L_{\mathbb{F}_p}^{\mathrm{alg}} \longrightarrow L_R^{alg} \longrightarrow L_{R/\mathbb{F}_p}^{alg}$ shows that we need only show that $L_{R/\mathbb{F}_p}^{alg}$ is an almost perfect R-module.

Let $R^{1/p}$ denote R thought of as an R-algebra via the Frobenius map. The F-finiteness of R implies that $L_{R^{1/p}/R}$ is an almost perfect $R^{1/p}$-module (this uses that R and hence $R^{1/p}$ are coherent, so one can just check that the homotopy groups are finitely presented).

Now, there is a fiber sequence

$$R^{1/p} \otimes_R L_{R/\mathbb{F}_p}^{alg} \longrightarrow L_{R^{1/p}/\mathbb{F}_p}^{alg} \longrightarrow L_{R^{1/p}/R}^{alg}$$

The final term in this fiber sequence is almost finite. We want to show that the middle term is almost finite, because there is no difference between asking for the middle term to be an almost finite $R^{1/p}$-module and asking that $L_{R/\mathbb{F}_p}^{alg}$ be an almost finite R-module.

Our worry is that the map

$$R^{1/p} \otimes_R L_{R/\mathbb{F}_p}^{alg} \longrightarrow L_{R^{1/p}/\mathbb{F}_p}^{alg}$$

might have a large image in homotopy groups, causing much cancellation. However, we will prove below that this map is just the 0 map. It is in the proof that the map is 0 that we really use the trick of exchanging topological for algebraic cotangent complexes.

Considered as a simplical $\mathbb{F}_p$-algebras, any R is a geometric realization $|P_\bullet|$ of polynomial algebras $P_n = \mathbb{F}_p[x_1, x_2 \cdots]$, on possibly finitely and possibly infinitely many generators. In this way, Jacob reduces things to the calculation of, for each n, the map

$$\Omega_{P_n/\mathbb{F}_p} \longrightarrow \Omega_{P_n^{1/p}/\mathbb{F}_p}.$$

But $d(x^p) = px^{p-1} = 0$ for any $x \in P_n$.

To finish, I ought not to omit the following fact:

Theorem 2.6. *Let R be a Noetherian $\mathbb{F}_p$-algebra which is complete with respect to an ideal $I \subset R$. If R/I is F-finite, then R is F-finite.*

This makes it reasonable that, in Jacob's strategy of building a tower of approximations to an $\mathbb{E}_\infty$-ring, the F-finiteness condition is really on a condition on the base of the tower. Since I is not assumed finitely generated, this is one theorem where the word *Noetherian* is really highlighted, as opposed to merely *coherent*.

Proof. Since R is Noetherian, we may choose a collection $(x_1, \cdots, x_n)$ of elements generating I. Let J denote the ideal $(x_1^p, x_2^p, \cdots, x_n^p)$, so that we have $I^{p^n} \subseteq J \subseteq I$.

The strategy is to first check that, since R/I is F-finite, $(R/J)^{1/p}$ is also finitely generated as an R-module.

This lets us picking generating elements $y_1, y_2, \cdots, y_m$, so that, for each $t \in R$,

$$t \equiv c_{1,1}^p y_1 + \cdots c_{1,m}^p y_m$$

modulo J.

In fact, we can make a sequence of m-tuples of elements $c_{i,1}, c_{i,2}, \cdots, c_{i,m}$ such that

$$t \equiv c_{i,1}^p y_1 + \cdots c_{i,m}^p y_m$$

modulo J^i, and where $c_{i+1,m} \equiv c_{i,m}$ modulo I^i.

Since R is complete with respect to I, we obtain by completion an integral equation

$$t = c_1^p y_1 + \cdots c_m^p y_m.$$

□