The relative cotangent complex

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The idea is to do a more spectral version of what Jeremy talked about last time. We want to understand the deformation theory of a p-divisible group; by definition, this is a morphism $\operatorname{Spec} R_0 \to \mathcal{M}_{BT}$ (moduli of p-divisible groups). We would like to understand why there is a universal deformation, which corresponds in our context to a factorization

$$\begin{array}{ccc} \operatorname{Spec} R_0 & \longrightarrow & \mathcal{M}_{BT} \\ & \searrow \quad \nearrow & \\ & \operatorname{Spf} R^{un}_{\mathbb{G}_0} & \end{array}$$

Think about this as a formal neighborhood of $\operatorname{Spec} R_0$ inside $\mathcal{M}_{BT}$.

Here's a first pass at what I'll try to explain.

Theorem 1. *When $\mathcal{M}_{BT}$ has a good deformation theory and $L_{\operatorname{Spec} R_0/\mathcal{M}_{BT}}$ is 1-connective and almost perfect, then you get a factorization as above.*

A lot of the details surrounding "good deformation theory" are in Elliptic 1. For now, we'll try to focus on the role that the cotangent complex plays in the story. The almost perfect condition is satisfied in our case because of the F-finiteness assumption Jeremy mentioned last time - we'll see more on that next time.

Notation 2. If X is a spectrum, write $X \geq n$ if X is n-connective. If f is a map, write $f \geq n$ if $\operatorname{fib}(f) \geq n$.

The idea is to make a hopefully convergent sequence

$$\operatorname{Spec} R_0 \to \operatorname{Spec} R_1 \to \operatorname{Spec} R_2 \to \cdots \to \mathcal{M}_{BT}.$$

The toy example to keep in mind is $\operatorname{Spec} A/x \to \operatorname{Spec} A$. Then the sequence would be "successively fattening this up", i.e. $\operatorname{Spec} A/x \to \operatorname{Spec} A/x^2 \to \operatorname{Spec} A/x^4 \to \cdots \to \operatorname{Spec} A$. At each stage, the thing you're adding on is the module of differentials - in the spectral setting, this will be replaced with the cotangent complex.

1. Primer on square-zero extensions

Say R is a classical ring and $M \in \operatorname{Mod}_R$. Say $M \to \widetilde{R} \to R$ is a square-zero extension. Note $\widetilde{R}$ may or may not be trivial (if it's trivial $\widetilde{R}$ splits as $R \oplus M$). The example to keep in mind for the non-split case is $\mathbb{Z}/p^2 \to \mathbb{Z}/p$.

Disclaimer: These are notes I took during the lecture. I, not the speaker, bear responsibility for mistakes. If you do find any errors, please report them to: `<ekbelmont at gmail dot com>`.

Fact 3. The automorphisms in the category of rings with a map to R are given by

$$\mathrm{Aut}_R(\widetilde{R}) = \mathrm{Der}(R, M) = \mathrm{Hom}_R(\Omega_R, M) \tag{1}$$

Say R is an E_∞-ring. On the derived side, you have the (absolute) cotangent complex L_R, which has the universal property that $\mathrm{Map}_{E_\infty}(R, R \oplus M) = \mathrm{Map}_R(L_R, M)$. (Dylan thinks L is for "left-derived".) We want a space-level version of (1).

Dream 4. Square-zero extensions of R by $M = \mathrm{Map}_R(L_R, \Sigma M)$.

Depending on how you set up the theory, there might be some issues with this. However, you at least have a map $\mathrm{Map}(L_R, \Sigma M) \to$ square-zero extensions, sending $\eta : L_R \to \Sigma M$ to the square zero extension $\widetilde{R}$ classified by η. It will be given by the formula $\widetilde{R} = \mathrm{fib}(R \to R \oplus \Sigma M \to \Sigma M)$.

2. Connectivity of the cotangent complex

Let $f : A \to B$. We now study its relative cotangent complex $L_{B/A}$. I just want to recall one fact - that there is a cofiber sequence (of B modules)

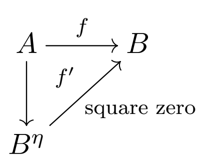
$$f^* L_A \to L_B \to L_{B/A}.$$

Question 5. In what sense does $L_{B/A}$ approximate the difference between A and B?

An example theorem we'd like to prove is: let $f : A \to B$ be a map of connective E_∞-rings. Then f is an equivalence iff $\pi_0 f$ is an isomorphism and $L_{B/A} \simeq 0$.

For this, we will need the following construction which explicitly compares the cofiber of f to the relative cotangent complex of f.

Construction 6. There's a natural map $\eta : L_B \to L_{B/A}$ classifying certain square-zero extensions of B; in particular, it classifies diagrams



This gives a map $\mathrm{cofib}(f) \to \mathrm{cofib}(f') = L_{B/A}$ of A-modules. Tensor this up to a map of B-modules to get $\varepsilon_f : B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$. This is the comparison map we'll be studying.

Here's a refined version of the example theorem:

Theorem 7. *Let $f : A \to B$ be a map of connective E_∞-rings. If $\mathrm{cofib}(f) \geq n$ then $\varepsilon_f : B \otimes_A \mathrm{cofib}(f) \to L_{B/A}$ is $2n$-connective.*

Proof. Say a map f is "n-good" if $\varepsilon_g \geq 2n$. We'll try to prove ε_f is n-good by making a filtration by killing free E_∞ cells. This is a good idea because the cotangent complex of a free E_∞ algebra is easy.

Construction: Suppose $f : A \to B$ satisfies $f \geq n-1$. The idea is to kill the bottom cell with an E_∞ cell to make it more connected.

$$\begin{array}{ccccc} \mathrm{fib}(f) & \longrightarrow & F_{E_\infty}(\mathrm{fib}(f)) & \longrightarrow & S \\ & \searrow & \downarrow & & \downarrow \\ & & A & \longrightarrow & A' \\ & & & \overset{f}{\searrow} & \vdots\, f' \\ & & & & B \end{array}$$

F_{E_∞} is the free E_∞ ring and A' is defined as the pushout. It's not too hard to see that $f' \geq n$, so I'll skip this.

Successively do this procedure and create a sequence

$$A = A_n \to A_{n+1} \to A_{n+2} \to \cdots \to A_{2n+1} \to B.$$

By virtue of the construction, this filtration has the following two properties:

(1) $\mathrm{cofib}(A_m \to B) \geq m$

(2) For all $m \geq n$, there exists $M \geq m-1$ such that there's a pushout in CAlg

$$\begin{array}{ccc} F_{E_\infty}(M) & \longrightarrow & S \\ \downarrow & & \downarrow \\ A_m & \xrightarrow{i_m} & A_{m+1}. \end{array}$$

In this situation, n-good morphisms turn out to be closed under composition (warning: not generally - you need some connectivity assumptions), so you reduce to showing to showing the following statements:

(1) $A_{2n+1} \to B$ is n-good.

(2) Each $A_m \to A_{m+1}$ is n-good (this is the key calculation)

The first of these is true just because the map is highly connected. We'll do the second one. It turns out that everything works well w.r.t. base change, and it suffices to show that $F_{E_\infty}(M) \to S$ is n-good when $M \geq n-1$.

$F_{E_\infty}(M)$ is free so its cotangent complex is just M. Running through our construction of the map ε_f, we get the triangle

$$\begin{array}{ccc} F_{E_\infty} & \xrightarrow{f} & S \\ & \searrow \quad \nearrow_{f'} & \\ & M \oplus S & \end{array}$$

which yields a map $\mathrm{cofib}(f) \to \mathrm{cofib}(f')$, which after tensoring up the source is the map

$$\varepsilon : \bigvee_{n\geq 1} (\Sigma M)^{\wedge n}_{h\Sigma_n} \to \Sigma M.$$

which is $2n$-connective because $M \geq n-1$ so $(\Sigma M)^{\wedge 2}_{h\Sigma_2} \geq 2n$. □

We can now prove the promised result.

Corollary 8. *Let $f : A \to B$ be connective E_∞-rings.*

(1) If $\mathrm{cofib}(f) \geq n$ then $L_{B/A} \geq n$.

(2) If $L_{B/A} \geq n$ and $\pi_0 f$ is an isomorphism, then $\mathrm{cofib}(f) \geq n$.

Proof. For (2), say $\mathrm{cofib}(f)$ is $(n-1)$-connective (i.e. not connective enough). $\pi_0 f$ is an isomorphism, so $n \geq 2$. From the theorem, $\varepsilon_f \geq 2n-2 \geq n$. Consider the following composite:

$$\pi_{n-1}(\mathrm{cofib}(f)) \to \pi_{n-1}(\mathrm{cofib}(f) \otimes_A B) \xrightarrow{\varepsilon_f} \pi_{n-1} L_{B/A}.$$

The first map is clearly an isomorphism, and the second map is an isomorphism by what we have just showed. The assumption about the connectivity of $\mathrm{cofib}(f)$ says that the first term is nonzero, and that implies the last term is nonzero, a contradiction. □

3. Enlarging a point in a stack to a formal neighborhood

Theorem 9 (Special case of SAG 18.2.5.1)**.** *Let B be an E_∞-ring and Y be a stack. Let $f : \mathrm{Spec}\, B \to Y$ be a natural transformation of functors $\mathrm{CAlg}^{cn} \to S$. Assume:*

(1) Y is nilcomplete ($Y(R) = \lim_n Y(\tau_{\leq n} R)$) and infinitesimally cohesive (analogue of saying there's patching—this is the thing in Jeremy's talk where if you have a surjective map of rings you get a pullback square) and admits a cotangent complex.

(2) Y is formally complete along f (this means $\mathrm{colim}\, \mathrm{Spec}\, B(R/I) \to \mathrm{colim}\, Y(R/I)$ is an equivalence, where the colimit is over all nilpotent ideals).

(3) f admits a relative cotangent complex $L_{\mathrm{Spec}\, B/Y}$ which is 1-connective and almost perfect.

Then Y is representable by an affine formal spectral Deligne-Mumford stack $\mathrm{Spf}(A)$.

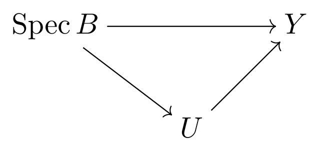
In the classical case, we had $\mathrm{Spec}\, B \to Y$ and created $\mathrm{Spf}(W(k)[[t_1, \ldots, t_n]])$ (free power series thing on the tangent space) that it factors through. We wanted to show that $\mathrm{Spf}(W(k)[[t_1, \ldots, t_n]]) \to Y$ is an equivalence; that's a formal consequence of both of these things satisfying (1) and (2), and the relative cotangent complex vanished. We're going to focus on producing this affine object. Classically, this relied on the tangent space being finite-dimensional; this is the analogue of (3). (1-connective has to do with smoothness, which is why you get an actual formal power series and not that mod something.)

Construction: Let's construct a sequence of formal schemes

$$\mathrm{Spec}\, B = \mathrm{Spec}\, B_0 \to \mathrm{Spec}\, B_1 \to \mathrm{Spec}\, B_2 \to \cdots \to Y.$$

Assume we have B_n. Do what we did in the first half: there's a map $L_{B_n} \to L_{\operatorname{Spec} B_n/Y}$ that classifies a square-zero extension. So you get a factorization $\operatorname{Spec} B_n \to \operatorname{Spec} B_{n+1} \to Y$. Define $\overline{U} = \operatorname{colim} \operatorname{Spec} B_n$. Define U so $U(R) = \lim_m \overline{U}(\tau_{\leq m} R)$ (forcing it to be nilcomplete). We'll eventually prove that U is affine.

You get a factorization



and:

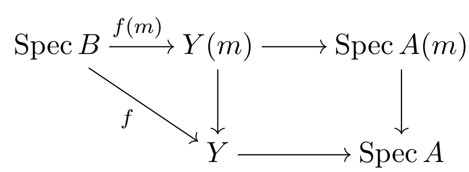
Fact 10. $U \simeq Y$

This is actually not the hard part. It's the analog of when we saw in Jeremy's talk that after producing a natural transformation between $\operatorname{Spf}(W(k)[[t_1, \ldots, t_n]])$ and the functor in question, it was automatically an equivalence by the formal deformation theoretic properties of the functors. The hard part is showing that $U \simeq Y$ is affine.

(Instead of writing U I'll write Y now.) $\Gamma(Y) = \varprojlim B_n =: A$ is a connective E_∞ ring with a natural map $f : A \to B_0 = B$. There's a global map of functors $Y \to \operatorname{Spec} A$ and the claim is that it factors through $\operatorname{Spf} A$, where I'm thinking of A as adic by the ideal $\ker \pi_0 f$. Y will be affine if and only if this is an equivalence; it should be believable at this point that what you need to see is the following proposition.

Proposition 11. *In this situation,* $L_{Y/\operatorname{Spec} A} \simeq 0$.

Strategy: The idea is to build up the analog of $\operatorname{Spf}(W(k)[[t_1, \ldots, t_n]])$ (from the classical situation) by successive approximation. We'll show that for each $m > 0$, there is an "m-good replacement" for $Y \to \operatorname{Spec} A$. An m-good replacement is the data of a diagram



where

1. the square is a pullback
2. $A(m)$ is almost perfect over A
3. the map $\pi_0 A \to \pi_0 A(m)$ is a surjection
4. $L_{\operatorname{Spec} B/Y(m)} \geq m$.

The $Y(m)$'s are supposed to be getting successively close to B.

Remark 12. If you have a replacement in the above sense, $\Gamma(Y(m)) \to A(m)$ is an equivalence. By definition, $Y(m) = \operatorname{Spec} A(m) \times_{\operatorname{Spec} A} \operatorname{colim} \operatorname{Spec} B_n$. We're in spaces, so we can pull out the colimit:

$$\Gamma(Y(m)) = \varprojlim A(m) \otimes_A B_n.$$

There's always a map $A(m) = A(m) \otimes_A \varprojlim B_n \to \varprojlim A(m) \otimes_A B_n$; the condition that $A(m)$ is almost perfect allows you to conclude that this map is an equivalence.

Lemma 13. *In this situation, suppose $L_{\operatorname{Spec} B/Y} \geq m$. Then for any $n \geq 0$,*

(1) $L_{\operatorname{Spec} B_n/Y} \geq m$

(2) $\operatorname{fib}(B_{n+1} \to B_n) \geq m - 1$

(3) $\operatorname{fib}(A \to B_n) \geq m - 1$

(4) $\theta : \operatorname{cofib}(A \to B) \to L_{B/A} \to L_{\operatorname{Spec} B/Y}$ *is surjective on π_m. (This is saying that this is surjective on the bottom homotopy group.)*

The first 3 claims are more or less immediate. For (4), look at

$$\begin{array}{ccc} & B_1 & \\ \nearrow & & \searrow \\ A & \longrightarrow & B = B_0 \end{array}$$

The claim is that analyzing how the diagram relates to θ shows $\operatorname{cofib}(\theta) = \operatorname{cofib}(A \to B_1) \geq m$.

Remark 14. We're trying to prove something about $L_{Y/\operatorname{Spec} A}$. But

$$L_{Y/\operatorname{Spec} A}\big|_{\operatorname{Spec} B} = L_{Y(m)/\operatorname{Spec} A(m)}\big|_{\operatorname{Spec} B}$$

because of the pullback for $Y(m)$ and properties of cotangent complexes. There's a fiber sequence

$$L_{Y/\operatorname{Spec} A}\big|_{\operatorname{Spec} B} = L_{Y(m)/\operatorname{Spec} A(m)}\big|_{\operatorname{Spec} B} \to L_{\operatorname{Spec} B/\operatorname{Spec} A(m)} \to L_{\operatorname{Spec} B/Y(m)}.$$

By part (3) plus the stuff we've prove, $L_{\operatorname{Spec} B/\operatorname{Spec} A(m)}$ is $\geq m$. By part (1), the last term is $\geq m$. Then we find the LHS is very connected. Since we can do this for every m,

$$L_{Y/\operatorname{Spec} A}\big|_{\operatorname{Spec} B} \simeq 0.$$

Since A is built as a nilpotent extension of B, it's a general fact (SAG 2.7.3.2) that this also implies

$$L_{Y/\operatorname{Spec} A} \simeq 0.$$

Let's now try to indicate how to prove the proposition.

Proof sketch for Proposition. Idea: proceed by induction; part (4) is telling you you can kill E_∞ cells on the bottom.

Suppose $m = 1$. Then $Y(1) = Y$ and $A(1) = A$ so it's OK.

Now assume we have it for m. Replace Y by $Y(m)$ and A by $A(m)$, so we can assume we're in the situation where $L_{\operatorname{Spec} B/Y} \geq m$ (i.e. where you can apply the last lemma). By assumption, $L_{\operatorname{Spec} B/Y}$ is almost perfect. This implies that π_m is finitely generated over $\pi_0 A$. We also have

$\mathrm{cofib}(A \to B) \to L_{B/A} \to L_{\mathrm{Spec}\, B/Y}$ is surjective on π_m. Choose a free A-module P and map $\Sigma^{m-1}P \to \mathrm{fib}(A \to B)$ such that $\Sigma^m P \to \mathrm{cofib}(A \to B) \to L_{\mathrm{Spec}\, B/Y}$ is surjective on π_m.

Define $A(m+1) := A \otimes_{F_{E_\infty/A}(\Sigma^{m-1}P)} A$ and $Y(m+1) = Y \otimes_{F_{E_\infty/A}(\Sigma^{m-1}P)} A$. You then have to show this works... $\square$