VERY ROUGH NOTES ON SPECTRAL DEFORMATION THEORY

1. Classical Deformation Theory

I want to begin with some classical deformation theory, before moving on to the spectral generalizations that constitute Lurie's work.

Definition 1.1. Let k be a field. A **formal moduli problem** is a functor

$$F : \mathbf{Art}_k \longrightarrow \mathbf{Groupoids}$$

such that $F(k) = *$. Here, $\mathbf{Art}_k$ is the category of local Artinian k-algebras and continuous homomorphisms.

Remark 1.2. The prototypical local Artinian k-algebra is the dual numbers $k[\epsilon]/\epsilon^2$. Any local Artinian k-algebra arises from k by a series of square-zero extensions. A more complicated example is

$$k[x, y]/(x^2, y^3, xy^2) \cong k \oplus kx \oplus ky \oplus kxy \oplus ky^2.$$

Example 1.3. Fix a finite height formal group law $f(x, y)$ over a perfect field of characteristic k. One can define a functor

$$Def_\star(-) : Art_k \longrightarrow \mathbf{Groupoids}$$

that takes A to the collection of pairs $(g(x, y), i)$, where $g(x, y)$ is a formal group law over A and i is a strict isomorphsim of the reduction of g with f.

The kind of theorem Lurie seeks to generalize is as follows:

Theorem 1.4. *(Lubin–Tate) There exists a complete local ring E, with residue field k, such that for any Artinian k-algebra A continuous maps from E into A represents the same functor as $\pi_0 Def_\star(A)$.*

Before discussing the generalization, let's briefly review how one might prove representability theorems classically. Here's a theorem with much too restrictive hypothesis, just to give you an idea of the kind of things involved.

Theorem 1.5. *(Schlessinger) Suppose that $F : Art_k \longrightarrow$ **Sets** is a **Sets** valued moduli problem, such as π_0 of some other moduli problem. Suppose that F satisfies the following criteria:*

1. *If $A', A'' \longrightarrow A$ are two surjections, then $F(A' \times_A A'') \simeq F(A') \times_{F(A)} F(A'')$.*
2. *F is formally smooth, meaning that for any surjection $A' \longrightarrow A$, the map $F(A') \longrightarrow F(A)$ is surjective.*
3. *The tangent space to F, $F(k[\epsilon]/\epsilon^2)$ is of finite dimension g.*

Then the functor F is pro-representable.

Remark 1.6. I should clarify what I mean when I say that $F(k[\epsilon]/\epsilon^2)$ has finite dimension g. After all, this tangent space is a priori just a set. The idea is to use the natural map $k \to \mathrm{End}_{\mathrm{Art}_k}(k[\epsilon]/\epsilon^2, k[\epsilon]/\epsilon^2)$ that sends a to $\alpha_a(x + y\epsilon) = x + ay\epsilon$. Applying F to α_a gives a candidate for scalar multiplication by a, that in fact forms a vector space structure.

Remark 1.7. Condition (1) is a gluing condition, and so is only reasonable when the set-valued fiber product agrees with the groupoid fiber product. In fact, $Def_\star(A)$ happens to be a discrete groupoid, so there is no real difference between $\pi_0(\mathrm{Def}_\star(A))$ and $\mathrm{Def}_\star(A)$. Many classical deformation problems are similarly discrete, such as deformations of elliptic curves or more general abelian varieties.

Proof of Schlessinger's theorem. Choose a basis $a_1, a_2, \cdots, a_g$ of $F(k[t]/t^2)$. Since F is formally smooth, we may lift each a_i to an element in $W(k)[[t]]/(p^2, t^2)$. In the limit, this gives elements

$$\alpha_i \in F(W(k)[[t]]) = \lim_n F(W(k)[[t]]/(p^n, t^n)).$$

Here, we **define** F on the non-Artinian $W(k)[[t]]$ by the filtered limit formula. Since F is product preserving, we get a single element

$$(\alpha_1, \alpha_2, \cdots) \in F(W(k)[[t_1, t_2, \cdots, t_g]]).$$

By Yoneda, that is the same data as a morphism of functors

$$\mathrm{Hom}(W(k)[[t_1, t_2, \cdots, t_g]], -) \longrightarrow F(-) \tag{1}$$

We would like to show that this map of functors is an equivalence, as it is by construction on $k[\epsilon]/\epsilon^2$. We prove that (1) is an equivalence on all A by induction on the length of A. To this end, suppose that A is an Artinian k-algebra, with a non-zero $x \in A$ annihilated by the maximal ideal in A, such that (1) is an equivalence when evaluated on A/x. We will show that (1) is an equivalence when evaluated on A.

Consider

$$\begin{array}{ccc} \mathrm{Hom}(W(k)[[t_1, t_2, \cdots, t_g]], A) & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(W(k)[[t_1, t_2, \cdots, t_g]], A/x) & \longrightarrow & F(A/x). \end{array}$$

We will show that the fibers of each vertical surjection of sets map bijectively onto one another.

To this end, consider the pull-back square

$$\begin{array}{ccc} F(A \times_{A/x} A) & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(A/x) \end{array}$$

Notice that $A \times_{A/x} A \cong A \times_k k[\epsilon]/\epsilon^2$, via the isomorphism that sends ϵ to x and A to the diagonal. The addition of x map thus exhibits A as a $(k[\epsilon]/\epsilon^2)$-torsor over A/x. Since F is product-preserving, and $F(A)$ surjects onto $F(A/x)$, it follows that $F(A)$ is naturally a torsor for $F(k[\epsilon]/\epsilon^2)$ over $F(A/x)$. The same is true for the functor $\mathrm{Hom}(W(k)[[t_1, t_2, \cdots, t_g]], -)$. □

2. Spectral deformations of p-divisible groups

Through the next few talks, we are going to greatly extend the above story in several directions:

1. We replace the field k with a more general Noetherian $\mathbb{F}_p$ algebra R_0.
2. We will replace the formal group law $f(x, y)$ with an arbitrary p-divisible group $\mathbb{G}_0$ over R_0.
3. We replace the category of local Artinian k-algebras with the much more general category of all adic $\mathbb{E}_\infty$-ring spectra.

Fix for the remainder of the talk a Noetherian $\mathbb{F}_p$-algebra R_0, as well as a p-divisible group $\mathbb{G}_0$ over R_0. What does it mean to give a deformation of $\mathbb{G}_0$ to an arbitrary adic $\mathbb{E}_\infty$-ring A?

Definition 2.1. Let A be an adic $\mathbb{E}_\infty$-ring and $\mathbb{G}$ a p-divisible group over A. A $\mathbb{G}_0$-tagging of $\mathbb{G}$ is a triple (I, μ, α), where $I \subset \pi_0(A)$ is an ideal of definition, $\mu : R_0 \longrightarrow \pi_0(A)/I$ is a ring homomorphism, and

$$\alpha : (\mathbb{G}_0)_{\pi_0(A)/I} \simeq \mathbb{G}_{\pi_0(A)/I}$$

is an isomorphism of p-divisible groups over the commutative ring $\pi_0(A)/I$.

A pair of $\mathbb{G}_0$-taggings (I, μ, α) and (I', μ', α') are equivalent if there exists a finitely generated ideal of definition $J \subset \pi_0(A)$, containing both I and I', for which the square

$$\begin{array}{ccc} R_0 & \longrightarrow & \pi_0(A)/I \\ \downarrow & & \downarrow \\ \pi_0(A)/I' & \longrightarrow & \pi_0(A)/J \end{array}$$

commutes, and α agrees with α' upon restriction to $(\mathbb{G}_0)_{\pi_0(A)/J}$.

Remark 2.2. Giving a $\mathbb{G}_0$-tagging of $\mathbb{G}$ is equivalent to giving a $\mathbb{G}_0$-tagging of the p-divisible group $\mathbb{G}_{\pi_0(A)/I}$, so this is all really a π_0 construction.

Definition 2.3. A *deformation* of $\mathbb{G}_0$ over an adic $\mathbb{E}_\infty$-ring A consists of a p-divisible group $\mathbb{G}$ over A together with an equivalence class of $\mathbb{G}_0$-taggings of A. This can be organized into an ∞-category

$$\mathrm{Def}_{\mathbb{G}_0}(A) = \mathrm{colim}_I \mathrm{BT}^p(A) \times_{\mathrm{BT}^p(\pi_0(A)/I)} \mathrm{Hom}(R_0, \pi_0(A)/I),$$

where I ranges over all finitely generated ideals of definition $I \subset \pi_0(A)$.

Remark 2.4. If $\pi_0(A)$ is Noetherian, then A admits a largest ideal of definition $I \subset \pi_0(A)$, and so the above filtered colimit collapses.

Definition 2.5. Let $\mathbb{G}$ denote a deformation of $\mathbb{G}_0$ over an adic $\mathbb{E}_\infty$-ring R. We will say that $\mathbb{G}$ is a *universal deformation* if, for every **complete** adic $\mathbb{E}_\infty$-ring A, the extension of scalars map

$$\mathrm{Map}_{CAlg^{ad}_{cpl}}(R, A) \longrightarrow \mathrm{Def}_{\mathbb{G}_0}(A)$$

is an equivalence.

Remark 2.6. This universal property guarantees that a universal deformation $(R, \mathbb{G})$, if it exists, is unique up to contractible choice.

We do not expect a universal deformation to always exist. For example, we need an analogue of the classical condition that the field k be perfect.

Definition 2.7. The $\mathbb{F}_p$-algebra R_0 is F-finite if the Frobenius homomorphism $\phi : R_0 \to R_0$ makes R_0 into a finitely presented R_0-module.

That's the analogue of a field k being perfect. Now, if R_0 is much bigger than a field, we can think of the formal group $\mathbb{G}_0$ as already being a sort of deformation of its reductions at the residue fields of R_0. We need to know that this R_0 deformation is in some sense as non-trivial as possible, and that we haven't already ruined the universality of our deformation.

Construction 2.8. Suppose $x \in |\mathrm{Spec}(R_0)|$ and d is a derivation

$$d : R_0 \longrightarrow \kappa(x),$$

where $\kappa(x)$ denotes the residue field of R at x. Then the projection $\beta_0 : R_0 \longrightarrow \kappa(x)$ extends to a map

$$\beta : R_0 \longrightarrow \kappa(x)[\epsilon]/(\epsilon^2),$$

given by the formula $\beta(t) = \beta_0(t) + \epsilon dt$. Let $\mathbb{G}_d$ be obtained by $\mathbb{G}_0$ via extension of scalars along β. We say that $\mathbb{G}_d$ is trivial iff it is isomorphic to extension along the composite

$$\beta_0 : R_0 \longrightarrow \kappa(x) \longrightarrow \kappa(x)[\epsilon]/(\epsilon^2).$$

Definition 2.9. The p-divisible group $\mathbb{G}_0$ is **non-stationary** if, for all choices of $\kappa(x)$ and non-zero derivations d, $\mathbb{G}_d$ is non-trivial

Example 2.10. Suppose that R_0 is semiperfect, meaning that the Frobenius $\phi : R_0 \to R_0$ is surjective. Then, for every R_0-module M, every derivation $d : R_0 \to M$ is trivial. This is just because $d(x^p) = px^{p-1}dx = 0$. Thus, in this case $\mathbb{G}_0$ is automatically non-stationary.

We can state the main theorem of the next few talks

Theorem 2.11 (Lurie). *Let R_0 denote an F-finite Noetherian $\mathbb{F}_p$-algebra and $\mathbb{G}_0$ a non-stationary p-divisible group over R_0. Then there is a univeral deformation $(R, \mathbb{G})$, where $R = R^{un}_{\mathbb{G}_0}$ is connective and Noetherian. Furthermore, the natural map $R \longrightarrow R_0$ induces a surjective homomorphism $\pi_0(R) \to R_0$, with kernel an ideal of definition for R.*

To apply a Schlessinger's criterion argument, we need to roughly:

(1) Know some 'formal smoothness' property for $\mathrm{Def}_{\mathbb{G}_0}(-)$.
(2) Obtain a complete understanding of the 'tangent space' to $\mathrm{Def}_{\mathbb{G}_0}$. We ought not just understand deformations over $k[\epsilon]/\epsilon^2$, but also deformations in 'derived directions'. We need to show that this tangent space is in some sense finite.

I won't talk about formal smoothness much in this talk, but maybe we'll talk about it in future talks. I will try to explain first a bit about (2), which is where the condition of F-finiteness becomes necessary.

The formal group law $\mathbb{G}_0$ is given by a map $\mathrm{Spec}(R_0) \longrightarrow \mathcal{M}_{BT}$. The notion of tangent space is replaced with the formalism of the relative cotangent complex

$$L_{\mathrm{Spec}(R_0)/\mathcal{M}_{BT}}.$$

We will want to talk about the thoery of cotangent complexes in more detail in future talks. Let me get started now, and eventually move to explain why the F-finiteness condition is relevant.

3. Cotangent Complexes of Rings

Suppose that A is an $\mathbb{E}_\infty$-ring spectrum. Then its cotangent complex L_A is an A-module with the following universal property

For any A-module M,

$$\mathrm{Der}(A, M) = \mathrm{Hom}_{\mathbb{E}_\infty-\mathrm{rings}/A}(A, A \oplus M) \simeq \mathrm{Hom}_{\text{A-mod}}(L_A, M)$$

This is a categorification of the notion of Kahler differential, which captures the usual cotangent space in algebraic geometry.

If $B \to A$ is a map of $\mathbb{E}_\infty$-ring spectra, there is also a relative version

$$\mathrm{Hom}_{A-modules}(L_{A|B}, M) \simeq \mathrm{Hom}_{\mathbb{E}_\infty-rings^{B/}_{/A}(A, A\oplus M)}$$

We want to know when a certain (relative) cotangent complex has a finiteness property. The correct notion of finiteness is 'almost perfection'.

Definition 3.1. Suppose that R is an $\mathbb{E}_\infty$-ring. Then a perfect R-module is a compact object in the category of R-modules. The perfect R-modules are those generated from finitely many copies of R under shifts, extensions, and retracts.

Example 3.2. Suppose that R is a classical Noetherian commutative ring, and that M is a classical finitely generated R-module. Then M admits a resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M,$$

where each P_i is free and finitely generated over R. However, M is only perfect if we can take the P_i to be 0 for $i \gg 0$. This is not always the case.

Definition 3.3. If R is a coherent $\mathbb{E}_\infty$-ring spectrum (meaning that R is connective, $\pi_0(R)$ is coherent, and each $\pi_n(R)$ is finitely presented), the category of almost perfect R-modules is the smallest one closed under retracts, shifts, finite colimits, and containing any geometric realization $|P_\bullet|$ of finite rank free R-modules.

Example 3.4. $\mathbb{Z}$ is an almost perfect $\mathbb{S}$-module.

In a future talk, we will discuss the following:

Theorem 3.5 (Lurie)**.** *Let R be a classical Noetherian $\mathbb{F}_p$-algebra. Then R is F-finite if and only if the absolute cotangent complex L_R is almost perfect as an R-module. Furthermore, in the case that $\mathbb{G}_0$ is non-stationary, $L_{Spec(R)|\mathcal{M}_{BT}}$, is 1-connected and almost perfect.*