Spectral formal groups

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Here's an outline for the rest of the seminar. First I and Eva will talk about formal and p-divisible groups over an E_∞ ring R. Then Jeremy will talk about the spectral deformation theory of p-divisible groups (this is about the $R^{un}_{\mathbb{G}}$ from Jacob's talk). Finally, we'll turn to the essential idea in this paper of orientations of spectral formal groups, and study how this relates to objects we already know something about like Lubin-Tate theory. That will give us $R^{or}_{\mathbb{G}}$.

1. Prologue: adic E_∞-rings

Definition 1. An adic ring is a pair (R, τ) (where R is a ring and τ is a topology on R) such that τ is the I-adic topology for some finitely generated ideal $I \subset R$. Note that I is not determined by this data.

I is called an *ideal of definition* for this adic ring.

Definition 2. An adic E_∞ ring is a pair (R, τ) where R is an E_∞ ring and τ is a topology on $\pi_0 R$ making it into an adic ring (so it admits a finitely generated ideal of definition).

Let CAlg^{ad} be the category of adic E_∞ rings, where the morphisms are E_∞ ring morphisms that are continuous on π_0.

Definition 3. Let R be an E_∞ ring and M an R-module. Given $x \in \pi_0 R$, we say M is (x)-complete if $\varprojlim(\dots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M)$ is contractible.

Equivalently,

$$\mathrm{Ext}^1_{\pi_0(R)}(\pi_0 R[x^{-1}], \pi_k M) = \mathrm{Hom}_{\pi_0(R)}(\pi_0 R[x^{-1}], \pi_k M) = 0$$

for all k. (Proof of equivalence: apply the Milnor exact sequence, where the Ext^1 corresponds to the $\lim^1$ term.)

(Take-away: it's just a condition on homotopy.)

For a finitely generated ideal $I \subset \pi_0 R$, say that M is I-*complete* if it is (x)-complete for all $x \in I$. It turns out that this is true iff it is true for a set $\{x_i\} \subset I$ of generators.

There is a completion functor $M \mapsto \widehat{M_I}$, which is the left adjoint of the inclusion of the I-complete modules in all modules. It is symmetric monoidal if I-complete modules are given the I-completed tensor product.

Definition 4. Given an adic R-algebra A (where R is an E_∞ ring), define $\mathrm{Spf}(A)$ to be a functor $\mathrm{CAlg}_R \to S$ where $\mathrm{Spf}(A)(B) = \mathrm{Map}^{cts}_{\mathrm{CAlg}_R}(A, B)$ where $\pi_0 B$ is discrete. Spf defines a functor $\mathrm{CAlg}^{ad}_R \to \mathrm{Fun}(\mathrm{CAlg}_R, S)$.

This is not fully faithful—for example, $\mathrm{Spf}(A) \simeq \mathrm{Spf}(\widehat{A_I})$. However, it is fully faithful on complete adic R-algebras (see SAG 8.1.5).

2. Formal groups

We're going to think about formal groups in three ways, which will be relevant for different reasons.

A formal group over R is a functor

$$\widehat{\mathbb{G}} : \mathrm{CAlg}_R \to \mathrm{Mod}^{cn}_{\mathbb{Z}} \simeq \mathrm{Ab}(\mathcal{S})$$

where $\mathrm{Ab}(\mathcal{S})$ denotes abelian group objects in spaces. (Why connective? Grouplike E_∞ spaces are connective spectra, not all spectra.) Note we're enforcing strict commutativity in our group objects. They're all affine: the composition $\Omega^\infty : \mathrm{CAlg}_R \to \mathrm{Ab}(\mathcal{S}) \to \mathcal{S}$ can be represented as $\mathrm{Spf}(A)$. But we want to make some restrictions on A.

Classically, if $R = k$, then $A = k[[t_1, \dots, t_n]]$ where the ideal of definition is $(t_1, \dots, t_n)$. In topology, we usually confine ourselves to $n = 1$, but that doesn't make anything simpler.

More generally, over a discrete ring R, we want $A \cong R[[t_1, \dots, t_n]]$ only Zariski locally on $\mathrm{Spec}\, R$ (only after inverting some element). Whatever condition we want on a formal group, it should be checkable Zariski-locally.

How do you generalize the notion of a power series ring to the E_∞ case? If E is a 2-periodic E_∞ ring (like Lubin-Tate theory) then we want $\mathrm{Spf}\, E^{\mathbb{C}\mathrm{P}^\infty}$ to be a formal group. If we just choose some specific model of formal power series, it won't be broad enough to include this example. For example, we could take $R[\mathbb{N}] \otimes_R \dots \otimes_R R[\mathbb{N}] =: R[t_1, \dots, t_n]$ and complete it at $(t_1, \dots, t_n)$. We could ask for things that are isomorphic to this as an E_∞ ring, and that's too restrictive. It'll end up that everything in our actual definition is isomorphic to this as an E_1 ring, but we want to include many ways to extend that E_1 structure.

Over an E_∞-ring R, we want to let A be any adic R-algebra such that

$$\pi_* A \cong \pi_* R[[t_1, \dots, t_n]]$$

Zariski-locally on $\pi_0 R$.

Audience question: Why do we want to consider abelian group objects and not all grouplike E_∞ spaces in our definition of formal groups?

Well, let's consider the case of the multiplicative group (the formal case is similar). We could look at $GL_1(R)$, defined by

$$\begin{array}{ccc} GL_1(R) & \hookrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R \end{array}$$

This will be co-represented by $S\{t\}[t^{-1}]$, where $S\{t\}$ denotes the free E_∞ ring on one variable t. The issue is that this is not flat over the sphere spectrum. On the other hand, you can define $\mathbb{G}_m(R) := \mathrm{Map}_{E_\infty}(\mathbb{Z}, GL_1 R)$. This is co-represented by $S[t][t^{-1}]$. The homotopy of this is $\pi_*(S)[t^{\pm 1}]$, and that is flat over S. So $\mathbb{G}_m$ is a much better behaved version of the multiplicative group than GL_1 from the point of view of spectral algebraic geometry.

As promised, here are three ways of describing a formal group $\widehat{\mathbb{G}}$:

(1) as an adic R-algebra A with properties as above;

(2) $\mathrm{Hom}^{cts}_{\mathrm{Mod}_R}(A, R)$ (as a coalgebra); This is equivalent to (1) but has the advantage that you don't need to remember the topology.

(3) functor of points: $\widehat{\mathbb{G}} : \mathrm{CAlg}_R \to S$

These are related via the following diagram:

$$\begin{array}{ccc} (1) & \xleftrightarrow{\text{dual}} & (2) \\ {\scriptstyle \mathcal{O}_X}\nwarrow\searrow{\scriptstyle \mathrm{Spf}} & & \swarrow{\scriptstyle c\,\mathrm{Spec}} \\ & (3) & \end{array}$$

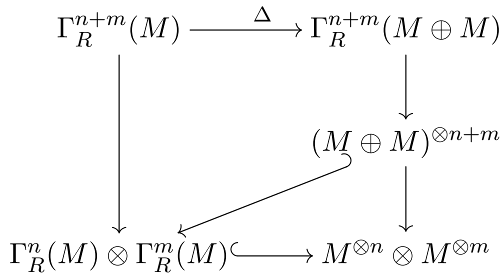
(1) has the advantage that it's identifiable in nature. For example, once I've computed $\pi_*(E^{\mathbb{C}P^\infty})$, I can check that $\mathrm{Spf}(E^{\mathbb{C}P^\infty})$ is a formal group. The coalgebra perspective is technically useful perspective for proving theorems about how these things work. (3) is the most conceptual and categorically nice perspective.

I'll run through the classical case of this diagram (where R is a discrete ring). Then we'll see it for the E_∞ case. From there we'll define formal groups formally, and using the work in the earlier sections we'll observe that we've already proved a bunch of nice properties about them.

3. Classical case

Let R be a discrete commutative ring and M a flat R-module.

Definition 5. Let $\Gamma^*_R(M) := \bigoplus_{n\geq 0} \Gamma^n_R(M)$ where $\Gamma^n_R(M) := (M^{\otimes_R n})^{\Sigma_n}$. We give this a coalgebra structure as follows: the co-unit $\Gamma^*(M) \to \Gamma^0_R(M) = R$ is the projection and the co-multiplication is given by summing up



If M is finitely generated projective (equivalently, Zariski-locally equivalent to R^n), then $\Gamma_R^*(M)$ is called a smooth coalgebra over R. Zariski-locally this is

$$\Gamma^*(R^n) = (R[[t_1, \dots, t_n]])^\vee$$

where the dual is defined in terms of continuous maps.

The part I want to focus on in this case is that the horizontal arrow (1) $\leftrightarrow$ (2) works. Let $(\Gamma_R^*(M))^\vee$ denote the R-linear dual. Using the fact that dualizing invariants is co-invariants, this is $\simeq \prod_{n \geq 0} \mathrm{Sym}_R^n(M^\vee)$. This is the completion $\widehat{\mathrm{Sym}^* R(M^\vee)}$ at $\mathrm{Sym}_R^{>0}(M^\vee) = I_M$. There's an issue—the topology is not defined canonically in terms of the coalgebra, but rather in terms of M. I want to show that you can get the topology canonically without reference to M.

Proposition 6. *The I_M-adic topology on $(\Gamma_R^*(M))^\vee$ is equivalent to the subspace topology induced by*

$$(\Gamma_R^*(M))^\vee \hookrightarrow \prod_{x \in \Gamma_R^*(M)} R = \mathrm{Map}_{\mathrm{Set}}(\Gamma_R^*(M), R)$$

where the right hand side is given the product topology.

Theorem 7. *Let $C, P \in c\,\mathrm{CAlg}_R^{sm}$ be smooth commutative coalgebras. Then dualization induces an isomorphism $\mathrm{Hom}_{c\,\mathrm{CAlg}_R}(D, C) \to \mathrm{Hom}_{\mathrm{CAlg}_R}^{cts}(C^\vee, D^\vee)$.*

The proof comes from using the universal property of divided power coalgebras.

4. E_∞ case

First I'll define what I mean by a coalgebra. Let C be a symmetric monoidal ∞-category. Define $c\,\mathrm{CAlg}(C) = \mathrm{CAlg}(C^{op})^{op}$ to be the ∞-category of commutative coalgebras.

Warning 8. A lax monoidal functor $C \to D$ does not induce a map $c\,\mathrm{CAlg}(C) \to c\,\mathrm{CAlg}(D)$. (You need the functor to be co-lax.) In particular, the functor $\pi_0 : \mathrm{Mod}_R^{cn} \to \mathrm{Mod}_{\pi_0 R}^\heartsuit$ is lax symmetric monoidal. (Jacob writes $\heartsuit$ for discrete, because it's the heart of a t-structure.) I.e. π_0 of a coalgebra is not a coalgebra. This is annoying because we want to define things in terms of π_0. Fix this by restricting to flat coalgebras, i.e. $\pi_0 M$ is flat over $\pi_0 R$ and

$\pi_n R \otimes_{\pi_0 R} \pi_0 M \to \pi_n M$ is an equivalence. Then π_0 above is symmetric monoidal on the nose becuase the Künneth spectral sequence collapses.

Let $\mathrm{Mod}_R^\flat$ denote flat R-modules. Then define

$$c\,\mathrm{CAlg}_R^\flat := c\,\mathrm{CAlg}(\mathrm{Mod}_R^\flat).$$

Then $c\,\mathrm{CAlg}_R^{sm}$ is the full subcategory of C such that $\pi_0 C$ is smooth over $\pi_0 R$.

Here are some important, and not-too-hard-to-prove facts (using facts about modules etc. and working on π_0):

(1) There is a canonical equivalence $- \otimes_{\tau_{\geq 0} R} R : c\,\mathrm{CAlg}_{\tau_{\geq 0} R}^\flat \leftrightarrows c\,\mathrm{CAlg}_R^\flat : \tau_{\geq 0}$.

(2) If R is connective, then *nilcompleteness* holds: that is, the map $c\,\mathrm{CAlg}_R^\flat \to \varprojlim c\,\mathrm{CAlg}_{\tau_{\leq n} R}^\flat$ is an equivalence of ∞-categories.

(3) (Clutching/ cohesiveness) Given a pullback of E_∞-rings

$$\begin{array}{ccc} A & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A_{01} \end{array}$$

then $c\,\mathrm{CAlg}_A \to c\,\mathrm{CAlg}_{A_0} \times_{c\,\mathrm{CAlg}_{A_{01}}} c\,\mathrm{CAlg}_{A_1}$ is fully faithful. Furthermore, if $\pi_0 A_0 \to \pi_0 A_{01} \leftarrow \pi_0 A_1$ are surjective then $c\,\mathrm{CAlg}_{A^\flat} \to c\,\mathrm{CAlg}_{A_0}^\flat \times_{c\,\mathrm{CAlg}_{A_{01}}^\flat} c\,\mathrm{CAlg}_{A_1}^\flat$ is an equivalence. If $\pi_0 A \to \pi_0 A_0$ has nilpotent kernel then $c\,\mathrm{CAlg}_A^{\mathrm{sm}} \to c\,\mathrm{CAlg}_{A_0}^{\mathrm{sm}} \times_{c\,\mathrm{CAlg}_{A_{01}}^{\mathrm{sm}}} c\,\mathrm{CAlg}_{A_1}^{\mathrm{sm}}$ is an equivalence.

When you try to prove something about smooth coalgebras, you often first reduce to the connective case, and then to the truncated case, and from there you can build up inductively via square-zero extensions using the Postnikov tower.

Fact 9. *$R \to c\,\mathrm{CAlg}_R^{sm}$ satisfies étale descent.*

Now let's consider duality in the E_∞ case.

$\underline{\mathrm{Map}}_R : \mathrm{Mod}_R^{op} \times \mathrm{Mod}_R \to \mathrm{Mod}_R$ is lax symmetric monoidal. Thus we get an induced map on commutative algebras $c\,\mathrm{CAlg}_R^{op} \times \mathrm{CAlg}_R \to \mathrm{CAlg}_R$.

Definition 10. Define $C^\vee := \underline{\mathrm{Map}}_R(C, R)$.

If C is smooth then it's projective in an appropriate sense. Then you can show the spectral sequence computing its homotopy groups of $C^\vee$ degenerates, which implies that $\pi_0(C^\vee) = (\pi_0 C)^\vee$. This is an adic R-algebra.

Theorem 11. *Given $C, D \in c\,\mathrm{CAlg}_R^{sm}$,*

$$\mathrm{Map}_{c\,\mathrm{CAlg}_R}(D, C) \to \mathrm{Map}_{\mathrm{CAlg}_R^{ad}}(C^\vee, D^\vee)$$

is a homotopy equivalence. (Here the ^{ad} *in* CAlg_R^{ad} *means continuous maps.) This implies that dualization is fully faithful into adic objects.*

I wanted to say something about the proof of this, but it looks like I don't have time. The basic idea is that you reduce to the connective case, then to the case of each truncation, which you handle inductively by going up the Postnikov tower. You then reduce a bit more and everything ends up following from duality for perfect R-modules. The key is that duality is an anti-equivalence $(\mathrm{Mod}_R^{\mathrm{perf}})^{op} \to \mathrm{Mod}_R^{\mathrm{perf}}$ symmetric-monoidally. (Perfect means generated under finite colimits by free things.) This implies that the dualization functor $(c\,\mathrm{CAlg}_R^{\mathrm{perf}})^{op} \to \mathrm{CAlg}_R^{\mathrm{perf}}$ is an equivalence.

Now let's characterize the image of the dualization funtor on smooth coalgebras.

Proposition 12. *Let A be an adic R-algebra. TFAE:*

(1) $A \simeq C^\vee$ for $C \in c\,\mathrm{CAlg}_R^{sm}$;

(2) for M finitely generated projective,

$$\pi_*(A) \simeq \prod_{n\geq 0} (\mathrm{Sym}^n_{\pi_0 R}(M) \otimes_{\pi_0 R} \pi_* R)$$

(this would be a power series ring if I took M to be free).

Again, do reductions and go up the Postnikov tower. There's some stuff you need to check about completions working out.

I'll finish by defining coSpec and defining formal groups.

Let $C \in c\,\mathrm{CAlg}_R^\flat$ where R is connective. Then $c\,\mathrm{Spec}(C)$ is a functor $\mathrm{CAlg}_R^{cn} \to S$ such that $c\,\mathrm{Spec}(A)(A) = \mathrm{Map}_{c\,\mathrm{CAlg}_A}(A, A \otimes_R C)$. (If this were classical, it would just be group-like elements.)

Fact 13. $c\,\mathrm{Spec}(C) \simeq \mathrm{Spf}(C^\vee)$

Corollary 14. $c\,\mathrm{Spec} : c\,\mathrm{CAlg}_R^{sm} \to \mathrm{Fun}(\mathrm{CAlg}_R^{cn}, \mathcal{S})$ *is fully faithful.*

This follows from the fact that $C^\vee$ is complete and the aforementioned fact thatthat Spf is fully faithful on complete adic R-algebras, which is in SAG 8.1.5.

Definition 15. A *formal hyperplane over R* is a functor $X : \mathrm{CAlg}_R^{cn} \to \mathcal{S}$ in the essential image of $c\,\mathrm{Spec}$ on $c\,\mathrm{CAlg}_R^{sm}$.

Definition 16. A formal group over R is a functor $\widehat{\mathbb{G}} : \mathrm{CAlg}_R^{cn} \to \mathrm{Mod}_{\mathbb{Z}}^{cn} = \mathrm{Ab}(\mathcal{S})$ such that the composite $\mathrm{CAlg}_R^{cn} \to \mathrm{Ab}(\mathcal{S}) \to \mathcal{S}$ is a formal hyperplane.

There's a formal multiplicative group: take the multiplicative group I defined earlier and complete it at the identity setion. On the level of functors of points, this means that I take

$$\widehat{\mathbb{G}}_m(R) = \mathrm{fib}\left(\mathbb{G}_m(R) \to \mathbb{G}_m(R^{\mathrm{red}})\right),$$

where R^{red} is just the reduction of $\pi_0 R$.