

# OVERVIEW: ELLIPTIC COHOMOLOGY II

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Let's start with a review of the Hopkins-Miller theorem. Let  $k$  be a perfect field of characteristic  $p$ , and let  $\widehat{\mathbb{G}}_0$  be a 1-dimensional formal group of finite height  $n < \infty$  over  $k$ . Then Lubin and Tate showed  $\widehat{\mathbb{G}}_0$  has a universal deformation  $\widehat{\mathbb{G}}$  over a complete local Noetherian ring  $R_{LT}$  (the "Lubin-Tate ring") with residue field  $k$ . It's a deformation in the sense that when you reduce everything modulo the maximal ideal of  $R_{LT}$ , you get the formal group  $\widehat{\mathbb{G}}_0$  that you started with. And it's universal: any deformation in this sense over a complete local Noetherian ring  $R$  can be obtained from the universal deformation by some map  $R_{LT} \rightarrow R$  that's the identity on residue fields.

There is a non-canonical isomorphism

$$(1) \quad R_{LT} \cong W(k)[[v_1, \dots, v_{n-1}]].$$

(The Witt vectors  $W(k)$  sits canonically inside  $R_{LT}$  but nothing else is canonical.) Morava showed that this formal group satisfies Landweber's criterion, which implies that there exists an (essentially) unique even periodic cohomology theory  $E$  such that  $\pi_0 E \cong R_{LT}$ . Moreover, it's even periodic, and in particular complex orientable, and the associated formal group is the universal deformation:  $\widehat{\mathbb{G}} \simeq \mathrm{Spf}(E^0(\mathbb{C}P^\infty))$ .

This turns out to be better behaved than you'd expect. Landweber's criterion produces a cohomology theory, and that's represented by a spectrum with a homotopy commutative multiplication. A priori, you don't know this is  $E_\infty$  etc. You also don't know that this depends functorially on the information given: Lubin-Tate gives  $R_{LT}$  functorially from  $k$  and  $\widehat{\mathbb{G}}_0$ , and obtaining the cohomology theory is functorial, but Brown representability is not a functorial thing.

**Theorem 1** (Goerss-Hopkins-Miller).  *$E$  has an essentially unique  $E_\infty$ -structure, and depends functorially (as an  $E_\infty$  ring spectrum) on  $(k, \widehat{\mathbb{G}}_0)$ . In particular,  $E$  admits an action of  $\mathrm{Aut}(\widehat{\mathbb{G}}_0)$  (defined at the spectrum level and preserves the  $E_\infty$  structure on  $E$ ).*

This was originally proved using obstruction theory. You write down a moduli space for all  $E_\infty$  structures on  $E$  compatible with the homotopy commutative multiplication you start with. You write a spectral sequence to compute the homotopy groups of this moduli space. One of the pages of the spectral sequence is identically zero, which shows that the moduli space is contractible, and that's where the "essentially unique" bit comes from.

The paper you're studying is a generalization of this story. What if you start with a family of formal groups?

**Question 2.** What if we try to replace  $k$  by a commutative  $\mathbb{F}_p$ -algebra  $R_0$ ?

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**Disclaimer:** These are notes I took during the lecture. I, not the speaker, bear responsibility for mistakes. If you do find any errors, please report them to: <ekbelmont at gmail dot com>.

I claim this is a bad question. In families, formal groups don't have a good deformation theory: 1-dimensional formal groups are classified by a moduli stack

$$\mathcal{M}_{FG} \simeq (\mathbb{A}^\infty)/(\text{group of all coordinate changes}).$$

Here  $\mathbb{A}^\infty$  is the moduli space of formal group laws (it's an affine space of infinite dimension by Lazard's theorem). It doesn't make a lot of sense to talk about the dimension of  $\mathcal{M}_{FG}$ : both  $\mathbb{A}^\infty$  and the group of coordinate changes are infinite dimensional, and indeed the dimension of  $\mathcal{M}_{FG}$  varies across points. Each formal group over a perfect field of characteristic  $p$  gives rise to a point in the moduli stack, and each  $R_{LT}$  describes a formal neighborhood of that point. (1) says the formal neighborhood of a point corresponding to a height- $n$  formal group has dimension  $n$ .

A better question:

**Question 3.** What if we replace  $k$  by  $R_0$  and  $\widehat{\mathbb{G}}_0$  by a  $p$ -divisible group  $\mathbb{G}_0$  over  $R_0$ ?

**Definition 4** (Tate). Let  $R$  be a commutative ring. A  $p$ -divisible group over  $k$  is a functor  $\mathbb{G} : \{\text{commutative } R\text{-algebras}\} \rightarrow \{\text{abelian groups}\}$  such that:

- (1)  $\mathbb{G}(A)$  is  $p$ -power torsion group, i.e.  $\mathbb{G}(A) = \bigcup \mathbb{G}(A)[p^n]$  (here  $[p^n]$  means  $p^n$ -torsion).
- (2)  $p : \mathbb{G} \rightarrow \mathbb{G}$  is surjective locally in the flat topology: given  $x \in \mathbb{G}(A)$ , you're allowed to (non-uniquely) divide  $x$  by  $p$  in some faithfully flat extension of  $A$ .
- (3) For each  $n \geq 0$ , the functor  $A \mapsto \mathbb{G}(A)[p^n]$  is represented by a finite flat  $R$ -scheme.

**Example 5.** Suppose  $\widehat{\mathbb{G}}$  is a formal group over a field  $k$  and  $[p] : \widehat{\mathbb{G}} \rightarrow \widehat{\mathbb{G}}$  is finite flat. Then  $\widehat{\mathbb{G}}$  is a  $p$ -divisible group over  $k$ . Such  $\widehat{\mathbb{G}}$  are known as *connected  $p$ -divisible groups over  $k$*  (where  $k$  has characteristic  $p$ ).

You can also think of formal groups as functors from commutative  $R$ -algebras to abelian groups, and some of them might be  $p$ -divisible.

**Example 6.** Define  $\underline{\mathbb{Q}_p/\mathbb{Z}_p}(A) =$  the set of locally constant  $(\mathbb{Q}_p/\mathbb{Z}_p)$ -valued functions on  $\text{Spec}(A)$ . (The underline means I'm thinking of this as a sheaf.) More generally, one considers powers  $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ , where you replace " $(\mathbb{Q}_p/\mathbb{Z}_p)$ -valued" with " $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ -valued" in the definition. These are called *constant  $p$ -divisible groups*.

If  $k$  is algebraically closed of characteristic  $p$ , any  $p$ -divisible group  $\mathbb{G}$  splits canonically as  $\mathbb{G}_{\text{inf}} \oplus \mathbb{G}_{\text{const}}$ , where  $\mathbb{G}_{\text{inf}}$  comes from a formal group ("inf" means "infinitesimal") and  $\mathbb{G}_{\text{const}}$  is constant. More generally, you have a SES

$$0 \rightarrow \mathbb{G}_{\text{inf}} \rightarrow \mathbb{G} \rightarrow \mathbb{G}_{\text{ét}} \rightarrow 0$$

called the "connected-étale sequence," where  $\mathbb{G}_{\text{ét}}$  is étale (becomes constant when you pass to the algebraic closure of  $k$ ). The sizes of the two components can vary, which is why thinking about just the size of the formal group (the infinitesimal part) is not a good thing to do. For example, some elliptic curves have height 1 and some have height 2 (the supersingular ones); i.e. the dimension of the formal group part changes as you move around the moduli space of elliptic curves, but the  $p$ -divisible group story is nicer.

More generally, given a  $p$ -divisible group  $\mathbb{G}$  over  $\mathrm{Spec} R$ , for every point  $x \in \mathrm{Spec} R$ , you get a  $p$ -divisible group  $\mathbb{G}_x$  over the residue field  $\kappa(x)$ . If  $\kappa(x)$  has characteristic  $p$ , by the connected-étale sequence there's an associated formal group at that point, and the associated étale part at that point, which looks like  $(\mathbb{Q}_p/\mathbb{Z}_p)^n$  with a Galois action.

In general, there's a procedure for extracting a formal group from a  $p$ -divisible group.

Let  $R_0$  be a nice  $\mathbb{F}_p$ -algebra, and let  $\mathbb{G}_0$  be a nice 1-dimensional  $p$ -divisible group over  $R_0$ . Then  $\mathbb{G}_0$  has a universal deformation. That is, there exists a Noetherian ring  $R_{\mathbb{G}_0}^{\mathrm{cl}}$  (“classical deformation ring”) with a surjection  $\varepsilon : R_{\mathbb{G}_0}^{\mathrm{cl}} \rightarrow R_0$  such that  $R_{\mathbb{G}_0}^{\mathrm{cl}}$  is complete w.r.t.  $\ker(\varepsilon)$  (if  $R_0$  were a field, this would be saying it's a complete local ring) and a  $p$ -divisible group  $\mathbb{G}$  over  $R_{\mathbb{G}_0}^{\mathrm{cl}}$  such that

$$\mathbb{G} \times_{\mathrm{Spec}(R_{\mathbb{G}_0}^{\mathrm{cl}})} \mathrm{Spec}(R_0) \simeq \mathbb{G}_0,$$

and moreover  $(\mathbb{G}, R_{\mathbb{G}_0}^{\mathrm{cl}})$  is the universal thing with this property. ( $R_0$  is the analogue of  $k$  in the Lubin-Tate story.)

What did we mean by “nice”, above? Say  $R_0$  is nice if  $R_0$  is finitely generated as a module over  $(R_0)^p$ . We ask for a finite map  $\varphi_{R_0} : R_0 \rightarrow R_0$ .  $\mathbb{G}_0$  is nice if the “restriction of  $\mathbb{G}_0$  to any tangent vector in  $\mathrm{Spec}(R)$  is non-constant”. (This is called *non-stationary* in the paper. We're trying to avoid  $\mathbb{G}_0 \times X$  over  $\mathrm{Spec}(R_0) \times X$ , i.e. something “constant in the  $X$  direction”.)

**Theorem 7.** *In this situation, there is an even (weakly) periodic cohomology theory  $E$  such that  $E^0(pt) = R_{\mathbb{G}_0}^{\mathrm{cl}}$  and  $\mathbb{G}^\circ \simeq \mathrm{Spf}(E^0(\mathbb{C}P^\infty))$  where  $\mathbb{G}^\circ$  is the identity component. (So far, you can produce this from Landweber's theorem.) Moreover,  $E$  has a canonical  $E_\infty$ -structure (i.e. the space of such structures has a distinguished point) which depends functorially on  $(R_0, \mathbb{G}_0)$ .*

**Definition 8.** Let  $R$  be a commutative ring with  $p$  nilpotent and let  $\mathbb{G}$  be a  $p$ -divisible group over  $R$ . Then define

$$\mathbb{G}^\circ(A) = \ker(\mathbb{G}(A) \rightarrow \mathbb{G}(A^{\mathrm{red}})).$$

**Theorem 9** (Messing).  *$\mathbb{G}^\circ$  is a formal group.*

From this definition it's clear it commutes with base change. This is what was meant earlier by saying you can extract a formal group from a  $p$ -divisible group; but only over a field do you get that SES.

**Idea of proof of Theorem 7.** The proof is totally different from the Goerss-Hopkins-Miller approach. The idea is to try to repeat the definition of the classical deformation ring  $R_{\mathbb{G}_0}^{\mathrm{cl}}$  but in the setting of  $E_\infty$  ring spectra. This was a commutative ring complete w.r.t. a topology, which has a universal property—it's where the universal deformation of  $\mathbb{G}_0$  lives. The idea is that the spectrum version should admit a similar universal property in the realm of  $E_\infty$  ring spectra.

The first goal is to make sense of  $p$ -divisible groups and formal groups over ring spectra. You also want to make sense of the “take the identity component” construction.

We started with the classical story of thickening up  $\mathbb{G}_0$  over  $\mathrm{Spec} R_0$  to get  $\mathbb{G}^{cl}$  over  $\mathrm{Spec}(R_{\mathbb{G}_0}^{cl})$ . This is universal if you're interested in purely algebraic deformations, but if you're interested in studying deformations in ring spectra, there's a "more universal" one, which we call  $\mathbb{G}$  over  $\mathrm{Spec}(R_{\mathbb{G}_0}^{un})$ .

**Theorem 10.**  *$\mathbb{G}_0$  has a universal deformation in the setting of  $p$ -divisible groups over ring spectra. This universal deformation lives over  $E_\infty$  ring spectrum  $\mathbb{R}_{\mathbb{G}_0}^{un}$ .*

$R_{\mathbb{G}_0}^{un}$  is connective, Noetherian (all homotopy groups are finitely generated over  $R_{\mathbb{G}_0}^{cl}$ ), and  $\pi_0(R_{\mathbb{G}_0}^{un}) \cong R_{\mathbb{G}_0}^{cl}$  (this is basically the definition—when you restrict to ordinary rings, the new universal object has to satisfy the same universal property as the old one). Also we have  $(\pi_n R_{\mathbb{G}_0}^{un})[p^{-1}] \cong 0$ . Note that this is not the ring spectrum  $E$  in the theorem; it's connective as opposed to 2-periodic. In homotopy theory, there's a particular formal group we care about, coming from Quillen's theorem, and so far there's nothing here to relate to that.

Let's recall how formal groups are related to homotopy theory. Let  $E$  be a homotopy commutative, complex orientable, and 2-periodic ring spectrum. Then  $R = E^0(pt)$  is a commutative ring and the formal spectrum  $\widehat{\mathbb{G}}^{cl} := \mathrm{Spf} E^0(\mathbb{C}P^\infty)$  is a formal group over  $R$ . This is a construction due to Quillen. There's a variant of this: if  $E$  is an  $E_\infty$ -ring spectrum, then you can also make a formal group  $\widehat{\mathbb{G}} = \mathrm{Spf}(E^{\mathrm{CP}^\infty})$  (I mean the function spectrum).

**Definition 11.** Let  $\mathbb{G}$  be a  $p$ -divisible group over a  $p$ -complete  $E_\infty$ -ring  $R$ . An *orientation* of  $\mathbb{G}$  is an isomorphism  $\mathbb{G}^\circ \cong \mathrm{Spf} R^{\mathrm{CP}^\infty}$ . If  $R$  is not complex-oriented and 2-periodic, then there are no orientations.

**Proposition 12.** *Let  $\mathbb{G}$  be any  $p$ -divisible group over  $R$ . Then there is a universal map  $R \rightarrow R'$  such that  $\mathbb{G} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$  has an orientation.*

$R'$  is an  $R$ -algebra, such that if you want to map to any other  $R$ -algebra, that's the same as giving an orientation of  $R$  base-changed to that  $R$ -algebra.

Now we continue with the proof of Theorem 7. Let  $\mathbb{G}$  be the universal deformation of  $\mathbb{G}_0$ . This lives over  $\mathrm{Spec}(R_{\mathbb{G}_0}^{un})$ . Now apply Proposition 12 to  $(R_{\mathbb{G}_0}^{un}, \mathbb{G})$ , to get a universal map  $R_{\mathbb{G}_0}^{un} \rightarrow E$ ; moreover just as  $\mathbb{G}$  lives over  $R_{\mathbb{G}_0}^{un}$ ,  $\mathbb{G}_E = \mathbb{G} \times_{\mathrm{Spec} R_{\mathbb{G}_0}^{un}} \mathrm{Spec}(E)$  lives over  $E$ . Moreover,  $\mathbb{G}_E^\circ \simeq \mathrm{Spf} E^{\mathrm{CP}^\infty}$ . Over  $\pi_0 E$ , you also get an isomorphism  $\mathbb{G}_{\pi_0(E)}^\circ \cong \mathrm{Spf} E^0(\mathbb{C}P^\infty)$ .

We have a corresponding map  $R_{\mathbb{G}_0}^{cl} = \pi_0(R_{\mathbb{G}_0}^{un}) \rightarrow \pi_0 E$ . The main step in the proof of the theorem is showing that this is an isomorphism. You also need to compute  $\pi_1(E) \cong 0$ .

In the classical approach to this, you use Landweber exactness and Brown representability so you know the homotopy, but you don't know it's  $E_\infty$ . With this approach, you know it has good properties, but you don't know it's the thing you want—for example, you don't know what you've constructed isn't just zero. The hard work is in showing it's the right size.

Why is it the right size?

**Question 13.** How do you compute  $\pi_*E$ ?

Strategy: Use naturality properties of the construction. We started with  $\mathbb{G}_0 \rightarrow \text{Spec}(R_0)$ , and everything depends functorially on the  $p$ -divisible group you started with. For example, if I had a pullback

$$\begin{array}{ccc} \mathbb{G}_1 & \longrightarrow & \mathbb{G}_0 \\ \downarrow & & \downarrow \\ \text{Spec}(R_1) & \longrightarrow & \text{Spec}(R_0) \end{array}$$

I could attempt to run the same construction on the LHS, getting  $E'$  on the left and  $E$  on the right. You can use this to get information on  $\pi_*E$  from  $\pi_*E'$  and vice versa. You could start by taking  $R_1 = k$ . If you understand the resulting  $E'$ , you learn about  $E$  formally completed w.r.t. the ideal gotten by cutting out the point. This strategy allows you to reduce the problem to working over a field  $k$ . You can even reduce to the case where  $k$  is algebraically closed.

Earlier we said that there is a splitting  $\mathbb{G}_0 \cong \mathbb{G}_{\text{connected}} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^n$ . Recall  $E$  came from working with  $\mathbb{G}_0$  over  $k$ , but we can also do the construction with  $\mathbb{G}_{\text{connected}}$  instead of  $\mathbb{G}_0$ , getting a map  $E_{\text{connected}} \rightarrow E_{\mathbb{G}_0}$ . So without loss of generality  $R_0 = k$  is algebraically closed and  $\mathbb{G}_0$  is connected, i.e. it is a formal group. This is the situation we started the lecture with: we have

$$R_{\mathbb{G}_0}^{\text{cl}} = R_{LT} \leftarrow R_{\mathbb{G}_0}^{\text{un}} \rightarrow E$$

and the claim is that, in this case, the resulting  $E$  is the Lubin-Tate spectrum. To show this (this is not what's done in the paper), use your knowledge of the Lubin-Tate spectrum and show that this  $E$  satisfies the same universal property in the world of  $K(n)$ -local  $E_\infty$ -ring spectra. So in this case you can compute  $\pi_*E$  by checking that  $E$  is a Lubin-Tate spectrum. (Actually, you don't know yet that  $E$  is  $K(n)$ -local; so you can show  $L_{K(n)}E$  is the Lubin-Tate spectrum, and so you know its homotopy groups. It turns out that  $L_{K(n)}$  doesn't change the homotopy groups, but that requires some work.)

This approach gives a new proof of the Goerss-Hopkins-Miller theorem—you don't *need* to use this to show that  $E$  is the right thing. Idea: start with  $\mathbb{G}_0$  over  $\text{Spec}(k)$  as before. Construct  $R_{\mathbb{G}_0}^{\text{un}} \rightarrow E \rightarrow L_{K(n)}E$ . The latter thing has an easy universal property in the world of  $K(n)$ -local ring spectra. Mapping it to something involves a map  $R_{\mathbb{G}_0}^{\text{un}}$  to something, and it also has to factor through  $E$ . You started with a connected  $p$ -divisible group. The upshot that, in order to map to something, the data you need is on the level of  $\pi_0$ . More precisely:

**Proposition 14.** *If  $A$  is a  $K(n)$ -local  $E_\infty$  ring,  $\text{Hom}(L_{K(n)}E, A)$  is empty unless  $A$  is 2-periodic and complex oriented, and in that case it is the set of identifications of  $\text{Spf}(A^0(\mathbb{C}P^\infty))$  with deformations of  $\mathbb{G}_0$ .*

(Warning: I'm using *deformation* in a weird way— $A$  might not be local etc. A deformation over  $R$  means a  $p$ -divisible group over  $R$  and an ideal, and when you reduce modulo that ideal you get something that can be expressed as a base-change.) The classical construction of Lubin-Tate spectra calculates this, and gets the same universal property. (Specifically, they did it for  $A = L_{K(n)}E$ , but they didn't need this.) But you don't actually need that here, so

you get a new construction of Lubin-Tate spectra.

$$\mathrm{Hom}(L_{K(n)}E, A) \cong \mathrm{Hom}_A(A \wedge L_{K(n)}E, A)$$

**Remark 15.** You can use this universal property to compute  $\pi_*L_{K(n)}E$ . The strategy is to smash with (2-periodic) complex bordism: we have

$$L_{K(n)}MP \rightarrow L_{K(n)}(MP \wedge E) \leftarrow L_{K(n)}(E).$$

You can compute  $\pi_*$  of the middle thing in two ways: (1) use the second map to look at this as an  $E$ -algebra; this reduces to computing the  $E$ -homology of complex bordism; (2) use the first map, which behaves like it is étale. You have a universal property of  $E$  in the  $K(n)$ -local category and that gives a universal property of  $MP \wedge E$  as an  $MP$ -algebra. You can explicitly write down an  $MP$ -algebra with that universal property and compute the homotopy groups of that  $MP$ -algebra as an algebra over  $\pi_*MP$ . So the input to working out  $\pi_*L_{K(n)}E$  is the complex oriented homology of  $MP$  and the homotopy of  $MP$ —exactly what comes out of Quillen’s theorem.