

# CLASSICAL OBSTRUCTIONS AND $S$ -ALGEBRAS

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ABSTRACT. Classical obstruction theory can be applied to the problem of finding an  $S$ -algebra structure, or a commutative  $S$ -algebra structure, on a ring spectrum. It is shown that there is no obstruction to upgrading the homotopy unit in the ring spectrum to a strict unit in the  $S$ -algebra.

## 1. INTRODUCTION

A *ring spectrum* is a spectrum  $E$  equipped with a homotopy-associative multiplication map  $\mu : E \wedge E \rightarrow E$  which has a two-sided homotopy unit  $\eta : S \rightarrow E$ . It is a *commutative ring spectrum* if  $\mu$  is homotopic to  $\mu\tau$ , where  $\tau$  interchanges factors in  $E \wedge E$ . Thus the (commutative) ring spectra are the (commutative) monoids in the stable homotopy category.

We should like to replace the multiplication  $\mu$  by a strictly associative multiplication map in the general case; and by a strictly associative and commutative multiplication map in the case of a commutative ring spectrum. (The object  $E$  may be replaced by a weakly equivalent object in the process.) These are notions at the point set or model category level, and they make sense if the model category which we are using has a symmetric monoidal smash product. We work in the category of  $S$ -modules, which has this property. It would be possible to adapt the theory, making necessary modifications, to other symmetric monoidal model categories for stable homotopy theory or to other contexts such as differential graded objects in an abelian category.

We consider the associative case in §2. The strictly associative multiplication which we seek on the  $S$ -module  $E$  can equivalently be described as an action of the associative operad  $\mathcal{M}$ , given by a morphism  $\varphi : \mathcal{M} \rightarrow \text{End}(E)$  of topological operads. To construct the action  $\varphi$  we replace  $\mathcal{M}$  by a suitable cofibrant resolution (in the appropriate category of topological operads). Under our assumptions regarding units in the multiplicative theories, this is the Stasheff operad  $\mathcal{A}$  of associahedra. In §3 we describe in detail an obstruction theory for finding a morphism of topological operads  $\varphi : \mathcal{A} \rightarrow \text{End}(E)$ , beginning with the map  $\varphi_2$  which takes the one-point space  $\mathcal{A}_2$  to the point  $\mu \in \text{End}(E)_2 = \text{Map}(E \wedge E, E)$ . The vanishing of the obstructions suffices for  $E$  to be weakly equivalent to an  $S$ -algebra. This is a refinement of the theory described in [16]; by comparing the new theory with the old, we show in 3.12 that there is no obstruction to upgrading a homotopy unit to a strict unit when the multiplication is associative.

In §§4–5 we develop the corresponding obstruction theory for refining a commutative ring spectrum to a commutative  $S$ -algebra. This runs exactly parallel to the foregoing associative theory, except that the obstructions lie in the  $\Gamma$ -cohomology of the Hopf algebroid  $E_*E$  instead of the Hochschild cohomology. Our results here are refinements of those for the homotopy-unital case outlined in [17].

## 2. BACKGROUND TO THE ASSOCIATIVE CASE

Suppose that  $E$  is an  $S$ -module which is also a ring spectrum. Let  $\mathcal{M}$  be the topological operad governing associative multiplications. We work here without permutations (these are “non- $\Sigma$  operads”), so that every space  $\mathcal{M}_n$  has a single point. We denote by  $\text{End}(E)$  the operad with  $n$ th space  $\text{Map}(E^{(n)}, E)$ , where  $E^{(n)} = E \wedge E \wedge \cdots \wedge E$  is the  $S$ -module smash product of  $n$  factors. The structure which we should like to have on  $E$  is a morphism of non- $\Sigma$  operads  $\mathcal{M} \rightarrow \text{End}(E)$ , as this makes  $E$  into an  $S$ -algebra. We shall show that it is sufficient to construct this when  $\mathcal{M}$  is replaced by a cofibrant resolution. This apparently weaker requirement can be tackled by obstruction theory. We make essential use of properties of cofibrant operads. It is not really necessary for our purposes to formalize the model category concerned, because we only need mapping properties for specific examples where they can be verified simply and directly. However, the formalization can be done: a closed model structure on the category of operads has been described in the algebraic case by Hinich [9] and the model structure on topological operads can be defined in close analogy with [9, §6]. The fibrations (resp. weak equivalences) are the maps of operads which are fibrations (resp. weak equivalences) at each level.

The canonical cofibrant resolution of  $\mathcal{M}$  is the operad  $W\mathcal{M}$  of “plane trees with stumps” described in [5] and [10]. A morphism  $W\mathcal{M} \rightarrow \text{End}(E)$  corresponds to a multiplication on  $E$  which satisfies all higher associativity conditions and has a two-sided homotopy unit  $S \rightarrow E$  satisfying all expected coherence conditions.

The operad  $W\mathcal{M}$  is larger and freer than is necessary for our obstruction theory. (The situation resembles one in homological algebra, where one need not use a free resolution to calculate  $\text{Tor}$ , if a flat resolution is much simpler; nor need one use an injective resolution to calculate sheaf cohomology, as a flasque resolution will do.) The simplification here arises from the fact that it is unnecessary to investigate coherent homotopy units because strict units, which are better, are so easy to analyse. Let  $\mathcal{A}$  be the Stasheff operad of associahedra, described in detail below. We note that there is a factorization  $W\mathcal{M} \rightarrow \mathcal{A} \rightarrow \mathcal{M}$  of the cofibrant resolution  $W\mathcal{M} \rightarrow \mathcal{M}$ ; in the terminology of [4] the first map corresponds to making stumps ignorable. In plainer terms,  $\mathcal{A}$  represents  $A_\infty$ -structures with strict unit.

We need to examine the Stasheff operad  $\mathcal{A}$  in some detail. For  $n = 0$  and  $n = 1$  the space  $\mathcal{A}_n$  is a point, corresponding in these two cases respectively to the unit element and the identity map. For  $n \geq 2$  the space  $\mathcal{A}_n$  is a convex affine cell of dimension  $n - 2$ , and the composition maps  $\mathcal{A}_i \times \mathcal{A}_j \rightarrow \mathcal{A}_{i+j-1}$  are affine inclusions corresponding precisely to the inclusions of the top-dimensional faces of  $\mathcal{A}_{i+j-1}$ . Indeed  $\mathcal{A}_2$  is a point, representing a map specifying a multiplication of two factors. Next,  $\mathcal{A}_3$  is a line segment, representing an associativity homotopy between the maps represented by its two endpoints, which correspond to the substitutions of the multiplication  $\mathcal{A}_2$  for either of the two factors in that multiplication. Then  $\mathcal{A}_4$  is the famous Stasheff pentagon, in which the five edges correspond to substitutions of  $\mathcal{A}_2$  for variables in the 3-factor multiplication  $\mathcal{A}_3$ , or *vice versa*. The polytope  $\mathcal{A}_5$  is an affine 3-cell which has six pentagonal faces isomorphic to  $\mathcal{A}_4 \times \mathcal{A}_2$  or  $\mathcal{A}_2 \times \mathcal{A}_4$ , and three rectangular faces isomorphic to  $\mathcal{A}_3 \times \mathcal{A}_3$ ; and so on.

If  $E$  were an  $S$ -algebra, we should have a morphism of operads

$$\mathcal{A} \rightarrow \mathcal{M} \rightarrow \text{End}(E)$$

obtained by composing the  $\mathcal{M}$ -action with the resolution above. This composite is, philosophically speaking, the real homotopical nub of the algebra structure. The following proposition shows that an  $S$ -algebra can be recovered from it.

**Proposition 2.1.** *Let  $E$  be an  $S$ -module which is also a ring spectrum. Suppose that there is a morphism of operads  $\mathcal{A} \rightarrow \text{End}(E)$  under which the point  $\mathcal{A}_2$  is mapped to the given multiplication on  $E$ . Then  $E$  is weakly equivalent to an  $S$ -algebra.*

*Proof.* Using the cofibrancy of  $\mathcal{A}$ , we can construct an augmentation  $\mathcal{A} \rightarrow \mathcal{L}$  from the operad  $\mathcal{A}$  into the linear isometries operad  $\mathcal{L}$ , because the spaces  $\mathcal{L}_n$  are contractible. Now we can apply the non- $\Sigma$  variant of [8, II, Prop. 4.3] to replace the  $\mathcal{A}$ -spectrum  $E$  by a weakly equivalent non- $\Sigma$   $\mathcal{L}$ -spectrum. By [8, II, Props. 4.6 and 3.6] this yields an  $A_\infty$  ring spectrum, which can be converted (by smash product with  $S$ ) into a weakly equivalent  $S$ -algebra.  $\square$

### 3. OBSTRUCTION THEORY IN THE $A_\infty$ CASE

We now need to describe how obstruction theory can allow us to prove that the hypotheses of Proposition 2.1 can be satisfied. This will require some conditions on the homology theory represented by our ring spectrum  $E$ .

In order to simplify the algebra, we shall assume that  $E$  is homotopy commutative. We denote by  $R$  the graded coefficient ring  $\pi_*E$ , and by  $\Lambda$  the graded ring  $E_*E$ . Our assumption implies that these are both commutative; and  $\Lambda$  becomes an  $R$ -algebra by means of the homomorphism, conventionally denoted  $\eta_L$ , induced in homology by the unit map  $\eta : S \rightarrow E$ . The multiplication map on  $E$  induces an augmentation  $\Lambda \rightarrow R$ , so that  $\Lambda$  splits as a

$\Lambda$ -module into  $R \oplus \tilde{\Lambda}$ , where  $\tilde{\Lambda}$  is the quotient module  $\Lambda/R$  or the augmentation ideal of  $\Lambda$ .

**Definition 3.1.** The ring spectrum  $E$  satisfies the *perfect universal coefficient formula* if the following two conditions hold.

- (1) The algebra  $\Lambda$  is  $R$ -flat. Consequently  $E_*(Y \wedge E) \approx E_*Y \otimes_R \Lambda$  for every spectrum  $Y$ . By induction, the smash power  $E^{(n)}$  has  $E$ -homology  $\Lambda^{\otimes n}$ .
- (2) The natural map

$$E^*(E^{(n)}) \longrightarrow \mathrm{Hom}_R(E_*(E^{(n)}), R) \approx \mathrm{Hom}_R(\Lambda^{\otimes n}, R)$$

is an isomorphism for every  $n$ .

The first condition in 3.1 is satisfied by many ring spectra including all those representing Landweber exact homology theories. The second is more restrictive, but is true for a wide range of useful spectra (see [14]).

The second condition can be rewritten in a more convenient way. Assuming that 3.1(1) holds, the algebra  $\Lambda$  is the Hopf algebroid of  $E$ -homology co-operations, or dual Steenrod algebroid, and the homology of any spectrum  $Y$  is a  $\Lambda$ -comodule via a natural homomorphism  $E_*(Y) \longrightarrow E_*(Y) \otimes_R \Lambda$ . Using the cofreeness of the  $\Lambda$ -comodule  $\Lambda$ , we can write the condition in 3.1(2) as

$$3.1(3) \quad E^*(E^{(n)}) \approx \mathrm{Cohom}_\Lambda(E_*(E^{(n)}), E_*E) \approx \mathrm{Cohom}_\Lambda(\Lambda^{\otimes n}, \Lambda)$$

where  $\mathrm{Cohom}$  denotes homomorphisms of comodules.

**Standing hypothesis 3.2.** We assume henceforth throughout this paper that the ring spectrum  $E$  has a perfect universal coefficient formula: that is,  $E$  satisfies Definition 3.1; and that the map  $\eta : S \rightarrow E$  is a cofibration.

(It would be interesting to know whether the obstruction theory can be set up when these conditions are relaxed. There may well be a derived-category variant which works more generally.)

**Hochschild complexes 3.3.** We shall need two versions of the Hochschild cochain complex of  $\Lambda$  over  $R$ . Let  $C^{**}(\Lambda|R; R)$  be the standard *unnormalized* edition (with  $R$  as coefficients): thus

$$C^{m,*}(\Lambda|R; R) \approx \mathrm{Hom}_R^*(\Lambda^{\otimes m}, R)$$

where the second grading is the internal grading in the rings. The  $\Lambda$ -module structure on  $R$  is given by the augmentation  $\Lambda = \pi_*(E \wedge E) \rightarrow \pi_*E = R$ , and the formula for the coboundary  $\delta : C^{m,*}(\Lambda|R; R) \longrightarrow C^{m+1,*}(\Lambda|R; R)$  is

$$\begin{aligned} (\delta\theta)(\lambda_0 \otimes \lambda_1 \otimes \cdots \otimes \lambda_m) &= \lambda_0 \cdot \theta(\lambda_1 \otimes \cdots \otimes \lambda_m) \\ &\quad + \sum_{i=1}^m (-1)^i \theta(\lambda_0 \otimes \cdots \otimes \lambda_{i-1} \lambda_i \otimes \cdots \otimes \lambda_m) \\ &\quad + (-1)^{m+1} \theta(\lambda_0 \otimes \lambda_1 \otimes \cdots \otimes \lambda_{m-1}) \cdot \lambda_m. \end{aligned}$$

On the other hand the *normalized* Hochschild cochain complex of  $\Lambda$  over  $R$  is  $\tilde{C}^{m,*}(\Lambda|R; R) \approx \text{Hom}_R^*(\tilde{\Lambda}^{\otimes m}, R)$  where  $\tilde{\Lambda}$  is  $\Lambda/R$  (which is isomorphic to the augmentation ideal). The formula above still defines a coboundary, making  $\tilde{C}^{*,*}(\Lambda|R; R)$  a subcomplex of  $C^{*,*}(\Lambda|R; R)$ . By the Normalization Theorem, the inclusion is a weak equivalence, so that each complex has cohomology  $HH^{**}(\Lambda|R; R)$ .

**Definition 3.4.** An  $\hat{\mathcal{A}}_n$ -structure on the ring spectrum  $E$  is a collection of maps  $\mu_m : \mathcal{A}_m \rightarrow \text{End}(E)_m$  for  $2 \leq m \leq n$ , such that

- (1) the point  $\mathcal{A}_2$  is mapped by  $\mu_2$  to the multiplication in  $E$
- (2) the conditions for a morphism of operads are satisfied where defined.

The second clause in this definition means the following. Recall that the boundary of the  $(m - 2)$ -cell  $\mathcal{A}_m$  is a union of faces, each an embedded copy of  $\mathcal{A}_i \times \mathcal{A}_j$  where  $i + j = m + 1$ . The condition is that the restriction of  $\mu_m$  to each face must be the composite  $c \circ (\mu_i \times \mu_j)$ , where  $c$  is the corresponding composition in the operad  $\text{End}(E)$ . Note that we are here temporarily working with operads without unit, as was done in [16]. (The notation  $\hat{\mathcal{A}}$  is intended to suggest that something is omitted.) The homotopy unit is present in  $E$ , but is not part of the operad structure.

**The unit condition 3.5.** We have assumed that the given homotopy unit  $\eta : S \rightarrow E$  is a cofibration. Hence the wedge  $(S \wedge E) \vee (E \wedge S)$  is now included as a subspectrum in the smash product  $E \wedge E$  by the cofibration  $(\eta \wedge 1) \vee (1 \wedge \eta)$ . Since  $E$  is assumed to be an  $S$ -module, this wedge is isomorphic to  $E \vee E$ .

**Lemma 3.6.** *The given multiplication  $\mu_2 : E \wedge E \rightarrow E$  can be deformed by a homotopy to make its restriction into the folding map  $1_E \vee 1_E$ .*

*Proof.* By the homotopy extension property, it suffices to show that  $\mu_2|_{E \vee E}$  is homotopic to the folding map. However, it is not obvious that this condition is satisfied, because there is no reason why the left and right unit homotopies should agree on  $S \wedge S$ . Thus there appears to be an obstruction in  $\pi_1 E$ . This obstruction is in fact zero, by the same argument as proves that an  $H$ -space is always simple.  $\square$

Thus  $\eta$  can be assumed to be a strict unit for  $\mu_2$ .

We should like to have an  $\mathcal{A}_\infty$  structure in which  $\eta$  is a strict unit for all the maps  $\mu_n : \mathcal{A}_n \rightarrow \text{End}(E)_n$ . Let us consider what that means.

The Stasheff cells  $\mathcal{A}_n$  are related not only by face maps but also by degeneracy maps  $s_i : \mathcal{A}_n \rightarrow \mathcal{A}_{n-1}$  which are defined for  $1 \leq i \leq n$  and are related to the principal faces of  $\mathcal{A}_n$  very much as faces and degeneracies among simplices are related [20]. In terms of trees,  $s_i$  corresponds to pruning off the  $i$ th twig. In terms of operads, the  $s_i$  define the  $n$  operad compositions  $\mathcal{A}_0 \times \mathcal{A}_n \rightarrow \mathcal{A}_{n-1}$  with the one-point space  $\mathcal{A}_0$ , thus completing the operad  $\mathcal{A}$  to an operad with unit.

For  $1 \leq i \leq n$  there is a cofibration  $\eta_i : E^{(n-1)} \rightarrow E^{(n)}$  defined as the composite

$$E^{(n-1)} \approx E^{(i-1)} \wedge S \wedge E^{(n-i)} \xrightarrow{1^{(i-1)} \wedge \eta \wedge 1^{(n-i)}} E^{(n)}$$

We define the *large wedge*  $\bigvee^n E$  to be the union of the images of the  $\eta_i$  for  $1 \leq i \leq n$ : it is the  $S$ -submodule of points with at least one factor in  $S$ .

**Definition 3.7.** We say that  $\eta$  is a *strict unit* for  $\mu_n$  if the following diagram commutes for  $1 \leq i \leq n$

$$\begin{array}{ccc} \mathcal{A}_n & \xrightarrow{\mu_n} & \text{End}(E)_n \\ \downarrow s_i & & \downarrow \text{Map}(\eta_i, 1) \\ \mathcal{A}_{n-1} & \xrightarrow{\mu_{n-1}} & \text{End}(E)_{n-1} \end{array}$$

which can be interpreted by saying that a product of  $n$  factors is unaffected by a unit in the  $i$ th place among the arguments.

We note that the above condition just fixes the value of  $\mu_n$  on the image of  $\eta_i$  for each  $i$ . The relations among the degeneracy maps  $s_i$  imply that if  $\eta$  is a strict unit for  $\mu_{n-1}$  and for  $\mu_n$ , then Definition 3.7 prescribes the adjoint  $\mu'_n : \mathcal{A}_n \times E^{(n)} \rightarrow E$  uniquely and coherently on  $\mathcal{A}_n \times \bigvee^n E$ .

**Definition 3.8.** An  $\mathcal{A}_n$ -structure on the ring spectrum  $(E, \mu, \eta)$  is a  $\hat{\mathcal{A}}_n$ -structure (see 3.4) such that the map  $\eta : S \rightarrow E$  is a strict unit for  $\mu_m$ ,  $2 \leq m \leq n$ .

We are now ready to set up the obstruction theory in the associative case.

**Theorem 3.9.** *Let an  $\mathcal{A}_{n-1}$  structure  $\mu$  on  $E$  be given, where  $n \geq 3$  and  $E$  satisfies the conditions 3.2. Then the following hold.*

- (1) *There is an obstruction cocycle  $\tilde{\theta}_n(\mu)$  in the normalized Hochschild cochain group  $\text{Hom}_R^{3-n}(\tilde{\Lambda}^{\otimes n}, R)$  which vanishes if and only if  $\mu$  can be extended to an  $\mathcal{A}_n$  structure on  $E$ .*
- (2) *The Hochschild cohomology class  $[\tilde{\theta}_n(\mu)] \in HH^{n,3-n}(\Lambda|R; R)$  is zero if and only if the underlying  $\mathcal{A}_{n-2}$  structure on  $E$  can be extended to an  $\mathcal{A}_n$  structure.*

*Proof.* To extend  $\mu$  to an  $\mathcal{A}_n$  structure on  $E$ , we need only construct  $\mu_n : \mathcal{A}_n \rightarrow \text{End}(E)_n$ , or equivalently its adjoint  $\mu'_n : \mathcal{A}_n \times E^{(n)} \rightarrow E$ , in such a way as to be compatible with composition in the operads. There are two cases of this condition, and we must consider them separately. Preserving the compositions  $\mathcal{A}_i \times \mathcal{A}_n \rightarrow \mathcal{A}_{n+i-1}$  when  $i > 0$  means that  $\mu_n$  is already defined on the decomposable elements of  $\mathcal{A}_n$ , which form the boundary of

this  $(n - 2)$ -cell. Preserving the compositions  $\mathcal{A}_0 \times \mathcal{A}_n \rightarrow \mathcal{A}_{n-1}$ , which is the condition of strong unitality, means that  $\mu'_n$  is already defined on  $\mathcal{A}_n \times \bigvee^n E$ . In all, the condition fixes  $\mu'_n$  on  $\partial\mathcal{A}_n \times E^{(n)} \cup \mathcal{A}_n \times \bigvee^n E$ .

The obstruction to extending  $\mu'_n$  over  $\mathcal{A}_n \times E^{(n)}$  therefore lies in the group

$$E^{-1}((\mathcal{A}_n, \partial\mathcal{A}_n) \times (E^{(n)} / \bigvee^n E)) .$$

By the Künneth Theorem we know that

$$E_*(E^{(n)} / \bigvee^n E) \approx E_*((E/S)^{(n)}) \approx \tilde{\Lambda}^{\otimes n}$$

and the homology sequence of the pair  $(E^{(n)}, \bigvee^n E)$  splits.

Furthermore  $\mathcal{A}_n / \partial\mathcal{A}_n$  is an  $(n - 2)$ -sphere, so by the universal coefficient formula the above group becomes

$$\begin{aligned} E^{-1}((\mathcal{A}_n, \partial\mathcal{A}_n) \times (E^{(n)} / \bigvee^n E)) &\approx E^{-1}(S^{n-2} \wedge (E^{(n)} / \bigvee^n E)) \\ &\approx \text{Hom}_R^{3-n}(\tilde{\Lambda}^{\otimes n}, R) \\ &\approx \tilde{C}^{n,3-n}(\Lambda|R; R) . \end{aligned}$$

Therefore the obstruction is a normalized Hochschild cochain, as claimed. We denote it by  $\tilde{\theta}_n(\mu)$ . If  $\tilde{\theta}_n(\mu) = 0$ , then the homotopy classes of extensions  $\mu_n$  are enumerated by difference classes in  $\tilde{C}^{n,2-n}(\Lambda|R; R)$ .

Consider the effect upon the obstruction cochain  $\tilde{\theta}_n(\mu)$  of changing  $\mu_{n-1}$  while keeping the  $\mathcal{A}_{n-2}$  structure fixed. Making this change alters the already-specified map  $\mu'_n | (\partial\mathcal{A}_n \times E^{(n)} \cup \mathcal{A}_n \times \bigvee^n E)$  on  $\mathcal{A}_n \times \bigvee^n E$ , and on certain faces of  $\mathcal{A}_n$ . Altering  $\mu'_n$  on  $\mathcal{A}_n \times \bigvee^n E$  does not affect the obstruction cochain, because the homology sequence for the pair  $(E^{(n)}, \bigvee^n E)$  splits. To find the effect of altering  $\mu_n | \partial\mathcal{A}_n$  we consider the faces separately. On a face which is an embedded copy of  $\mathcal{A}_i \times \mathcal{A}_j$ , the restriction of  $\mu_n$  is determined by  $\mu_i \times \mu_j$ . Since we are changing only  $\mu_{n-1}$ , those affected are the two faces isomorphic to  $\mathcal{A}_2 \times \mathcal{A}_{n-1}$  and the  $n - 1$  faces isomorphic to  $\mathcal{A}_{n-1} \times \mathcal{A}_2$ . These faces correspond precisely to the  $n + 1$  terms (of two kinds) in the above formula for the Hochschild coboundary.

Let us briefly explain why these faces give precisely the terms in the Hochschild coboundary formula. For the first and last terms, this is verified most easily by using formula 3.1(3). For all the other terms in the Hochschild formula, it is obvious, apart from the sign. We have to verify that the signs alternate. This is forced by the fact that, as we show at the end of this proof, the geometrical obstruction must always be a cocycle. (The details of calculation need to be slightly elaborated in the lowest case  $n = 3$ , but the result is the same.)

It follows that changing  $\mu_{n-1}$  by a difference class  $\alpha \in \tilde{C}^{n-1,3-n}(\Lambda|R; R)$  has the effect of altering the obstruction cochain by  $\tilde{\theta}_n(\mu)$  by  $\delta\alpha$ .

Therefore the obstruction  $\tilde{\theta}_n(\mu)$  can be reduced to zero by altering  $\mu_{n-1}$  if and only if it is a Hochschild coboundary. To complete the proof of Theorem 3.9 we show that  $\tilde{\theta}_n(\mu)$  is always a cocycle. This is true because the coboundary of  $\theta_n(\mu)$  is, by the argument over faces used above, the obstruction to extending  $\mu$  over the boundary of the boundary of  $\mathcal{A}_{n+1}$ ; and this is an empty space.  $\square$

To apply 3.9 recursively, we need a  $\mathcal{A}_2$  structure to start the induction. This is provided by Lemma 3.6. In fact we only need the cohomology to be zero for  $n \geq 4$ , because the obstruction cohomology class for the existence of a  $\mathcal{A}_3$ -structure is always zero, as we shall see in Theorem 3.12 below. This means we can always choose the associativity homotopy  $\mu_3$  so that it is constant when one of the three factors is the unit. (It is not particularly easy to prove this by direct, bare-hands construction.)

**Corollary 3.10.** *If  $HH^{n,3-n}(\tilde{\Lambda}|R;R) = 0$  for all  $n \geq 4$ , then the ring spectrum  $E$  has an  $\mathcal{A}_\infty$  structure. By Proposition 2.1,  $E$  can thus be represented by an  $S$ -algebra.*

**Comparison with the homotopy-unital theory 3.11.** In [16] we developed a theory exactly parallel to the above, but without the strict unital condition. The given ring spectrum  $E$  has a homotopy unit, but now we do not regard the unit as part of the operad structure. In terms of trees, we allow no stumps. In this case one again builds the  $\hat{\mathcal{A}}_n$ -structure – that is, the higher associativity conditions – by induction on  $n$ . The obstructions lie in the unnormalized Hochschild cochain complex. The proof is a simplified version of the proof of 3.9.

A ring spectrum is homotopy associative, and therefore already has an  $\mathcal{A}_3$  structure. We can therefore begin a recursive application of our homotopy-unital analogue at  $n = 4$ .

This result was used in [16] to show every Morava  $K$ -theory at an odd prime has an  $\hat{\mathcal{A}}_\infty$  structure (indeed, has uncountably many such structures).

We therefore have two variants of the obstruction theory, which seem to be essentially equivalent: they just give rise to the normalized and non-normalized Hochschild complexes, which have the same homology. One therefore guesses that the existence of a strict unit is, homotopically speaking, no more of a restriction than the existence of a homotopy unit; a fact which is confirmed by the next theorem. (This is not too surprising. There is a close analogy with the theory of  $H$ -spaces, and it has long been known, for instance through quasifibration theory, that a connected associative  $H$ -space with homotopy unit is equivalent to a Moore loop space, which has a strict unit.)

**Theorem 3.12.** *Suppose  $E$  is a ring spectrum satisfying 3.2. Let  $3 \leq n \leq \infty$ , and suppose that  $E$  admits a  $\hat{\mathcal{A}}_n$  structure. Then*



- (1)  $E$  admits a  $\mathcal{A}_n$ -structure
- (2) if  $\mathcal{A}_n(E)$  (respectively  $\hat{\mathcal{A}}_n(E)$ ) denotes the space of  $\mathcal{A}_n$ -structures (respectively  $\hat{\mathcal{A}}_n$ -structures) on  $E$ , then the forgetful map  $\Phi : \mathcal{A}_n(E) \rightarrow \hat{\mathcal{A}}_n(E)$  is a weak homotopy equivalence.

*Proof.* We shall show that all the homotopy groups of the map  $\Phi$  are trivial. Specifically, if for any  $k \geq 0$  we have any maps  $f : \Delta^k \rightarrow \hat{\mathcal{A}}_r(E)$  and  $g : \partial\Delta^k \rightarrow \mathcal{A}_r(E)$  such that  $f|_{\partial\Delta^k} \simeq \Phi g$ , then  $g$  extends to a map  $\Delta^k \rightarrow \mathcal{A}_r(E)$ . Since we can take  $k = 0$ , this will prove (1) as well as (2).

We prove the claim by setting up obstruction theory like that in 3.9, proceeding step by step up the operad. At the  $r$ th stage, we have precisely the problem of extending, over the interior of a  $k$ -cell, a deformation of  $\mu'_r | (\mathcal{A}_r \times \bigvee^r E)$  into the map prescribed by the unitality condition. (By homotopy extension, the deformation over  $\mathcal{A}_r \times E^{(r)}$  then follows.) The obstruction is a cocycle in the  $k$ th suspension of the mapping cone of the standard cochain map from the normalized Hochschild complex  $\text{Hom}_R^*(\hat{\Lambda}^{\otimes*}, R)$  to the unnormalized one, because the  $E$ -homology of  $\bigvee^r E$  is the degenerate part of  $\Lambda^{\otimes r}$ . Since normalization does not affect cohomology, this mapping cone is contractible; so the cocycle is a coboundary, and the extension exists.  $\square$

#### 4. BACKGROUND TO THE COMMUTATIVE CASE: GAMMA HOMOLOGY

We now aim to prove theorems exactly analogous to 3.9 and 3.12 which will handle commutativity and associativity simultaneously. This will allow us to replace a commutative ring spectrum (which is an abelian monoid object in the stable homotopy category) by a commutative  $S$ -module (which is the equivalent at the point-set level) provided that certain cohomological obstructions vanish.

Whereas Hochschild cohomology of algebras has been known for 50 years and the Stasheff operad for 40 years, the cohomology theory for commutative algebras and the cofibrant resolution of the commutative operad  $\mathcal{C}$  needed here are recent developments. The cohomology theory is  $\Gamma$ -cohomology [18]. It is no longer very new: this homology for commutative algebras had its origins in different ideas developed independently by the author and by F. Waldhausen in the late 1980's. A related theory, called topological André-Quillen cohomology, was invented for essentially the same purpose by Basterra [2] and Kriz. The dual homology theory also arises as an instance of Schwede's stable homotopy of algebraic theories [19]. The relations among all these different approaches are surveyed by Basterra and Richter [3].

We need a commutative ground ring for our homological algebra. With a view to the application, we denote it by  $\Lambda$ . There is no restriction on the characteristic. From our point of view, the homology involves the Lie representations of the symmetric groups. The connection of these with geometry will become apparent in 5.6.

**The Lie representations 4.1.** Let  $\mathcal{L}_n$  be the free Lie algebra over  $\Lambda$  on the set of generators  $\{x_i\}_{1 \leq i \leq n}$ . We denote by  $\text{Lie}_n$  the so-called *multilinear part* of  $\mathcal{L}_n$ . This can be described in many different ways. First, it is defined as the direct summand of  $\mathcal{L}_n$  spanned by all Lie monomials containing each of the  $n$  generators exactly once. Second, it is the  $n$ th module in the Lie operad. Third, it is isomorphic to the module of all natural transformations  $\Phi^{\otimes n} \rightarrow \Phi$ , where  $\Phi$  is the forgetful functor from Lie algebras to  $\Lambda$ -modules.

The symmetric group  $\Sigma_n$  acts upon  $\text{Lie}_n$  by permuting the  $n$  generators. The  $\Sigma_n$ -module thus obtained is known as the *Lie representation*. We twist it by the sign character, so that the left action of  $\Sigma_n$  on the abelian group  $\text{Lie}_n$  is defined by setting

$$\sigma \cdot f(x_1, \dots, x_n) = \varepsilon(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for every multilinear Lie monomial  $f$  and every  $\sigma \in \Sigma_n$ , where  $\varepsilon(\sigma)$  is the sign of  $\sigma$ . Let  $\text{Lie}_n^*$  be the dual module  $\text{Hom}(\text{Lie}_n, \Lambda)$ , which is thus a right  $\Sigma_n$ -module.

We shall require the following properties of  $\text{Lie}_n$  (see [21, 2.3]):

- (1) the left regulated Lie brackets

$$\sigma \cdot [x_1, [x_2, [x_3, \dots, [x_{n-1}, x_n] \dots]]] \quad \text{for } \sigma \in \Sigma_{n-1}$$

form a  $\Lambda$ -basis of  $\text{Lie}_n$ . Therefore

- (2) the  $\Lambda$ -modules  $\text{Lie}_n$  and  $\text{Lie}_n^*$  are free of rank  $(n-1)!$ , and  
(3) the restricted  $\Sigma_{n-1}$ -modules  $\text{Res}_{\Sigma_{n-1}}^{\Sigma_n} \text{Lie}_n$  and  $\text{Res}_{\Sigma_{n-1}}^{\Sigma_n} \text{Lie}_n^*$  are respectively isomorphic to the left and right regular representations.

**The  $\Xi$ -complex and stable homotopy 4.2.** Let  $\Gamma$  be the category of finite based sets, and  $[n]$  the typical object  $\{0, 1, \dots, n\}$  with 0 as the basepoint. A *left  $\Gamma$ -module* is a functor  $\Phi$  from  $\Gamma$  to  $\Lambda$ -modules. Such a functor converts simplicial finite sets into simplicial modules. Bousfield and Friedlander [6], developing ideas of G.B. Segal, show that the homotopy groups  $\pi_{n+i}\Phi(S^i)$  are independent of the simplicial model of the sphere  $S^i$ , and indeed independent of  $i$  for  $i > n$ . One therefore defines

$$\pi_n \Phi = \pi_{n+i} \Phi(S^i) \quad \text{for } i > n.$$

This result was originally proved for the more general case of  $\Gamma$ -spaces, but we now specialize to  $\Gamma$ -modules. A *right  $\Gamma$ -module* is a cofunctor from  $\Gamma$  to  $\Lambda$ -modules.

Let  $\Omega$  be the category of unbased finite sets  $\underline{n} = \{1, 2, \dots, n\}$ ,  $n \geq 0$ , with surjective maps as morphisms. Adding a disjoint basepoint defines an inclusion functor  $\Omega \rightarrow \Gamma$  taking  $\underline{n}$  to  $[n]$ . We regard the morphisms in the image of this functor, and other surjections, as face operators in  $\Gamma$ , and strict injections as degeneracy operators. There are additive categories  $\Lambda\Gamma$  and  $\Lambda\Omega$  with the same objects, indexed by non-negative integers, as  $\Gamma$  and  $\Omega$ , but having as morphisms the free  $\Lambda$ -modules generated by the morphism-sets of  $\Gamma$  or  $\Omega$ . We

regard these additive categories as rings with many objects. Pirashvili has shown [13] that there is a Morita equivalence between the categories  $\Lambda\Gamma\text{-mod}$  and  $\Lambda\Omega\text{-mod}$ . Indeed, let  $J$  be the complement  $\Gamma \setminus \Omega$ ; that is, the subset of morphisms  $\varphi \in \Gamma$  which satisfy  $\varphi^{-1}0 \neq \{0\}$ . Then  $\Lambda J$  is a  $(\Lambda\Gamma, \Lambda\Omega)$ -submodule of  $\Lambda\Gamma$ ; and the promised Morita equivalence  $\Lambda\Omega\text{-mod} \rightarrow \Lambda\Gamma\text{-mod}$  is given by tensoring on the left with the quotient  $(\Lambda\Gamma, \Lambda\Omega)$ -module  $\Lambda\Gamma/\Lambda J$ . The inverse equivalence is the *cross-effect functor*

$$cr : \Lambda\Gamma\text{-mod} \longrightarrow \Lambda\Omega\text{-mod}$$

given by an idempotent in  $\Lambda\Gamma$  which kills non-surjective morphisms of  $\Gamma$ . For the categories of right modules there is a dual situation.

Let  $t$  be the right  $\Lambda\Gamma$ -module  $\text{Hom}_{\text{Sets}_*}(-, \Lambda)$ . We denote  $\text{Tor}$  of  $\Lambda\Gamma$ -modules by  $\text{Tor}^\Gamma$  and  $\text{Tor}$  of  $\Lambda\Omega$ -modules by  $\text{Tor}^\Omega$ . The following is proved in [13, 2.2].

**Theorem 4.3.** (*Pirashvili*) *There is a natural isomorphism for  $\Lambda\Gamma$ -modules  $\Phi$*

$$\pi_*\Phi \approx \text{Tor}_*^\Gamma(t, \Phi) . \quad \square$$

If we take  $\Theta$  to be the cofunctor  $t$  of 4.3, then  $cr t$  is the unique  $\Omega$ -module  $\Theta$  such that  $\Theta(\underline{1}) = \Lambda$  and  $\Theta(\underline{n}) = 0$  for  $n \neq 1$ .

These  $\text{Tor}$ -groups can in turn be calculated from a certain bicomplex [17] called the  $\Xi$ -complex. It is based upon a projective resolution of the module  $t$ , constructed from the representations  $\text{Lie}_n^*$ .

**Theorem 4.4.** *Let  $\Phi$  be any  $\Gamma$ -module. There is a natural bicomplex  $\Xi(\Phi)$  in which the  $(q-1)$ st row is the two-sided bar construction  $\mathcal{B}(\text{Lie}_q^*, \Sigma_q, \Phi[q])$ , the vertical differential is induced by the Leibniz differential of [12] and the homology is*

$$H\Xi(\Phi) \approx \text{Tor}_*^\Gamma(t, \Phi) . \quad \square$$

The Morita equivalence converts the projective resolution of  $t$  into a projective resolution of the right  $\Omega$ -module  $\Theta = cr t$  described above. We therefore have:

**4.5.**

$$H\Xi(\Phi) \approx \pi_*\Phi \approx \text{Tor}_*^\Gamma(t, \Phi) \approx \text{Tor}_*^\Omega(\Theta, cr \Phi).$$

**The Loday functor and the  $\Gamma$ -homology of commutative graded algebras 4.6.** Let  $R = \{R_n\}_{n \in \mathbb{Z}}$  be an associative graded ring with unit which is commutative in the usual graded sense: that is,  $yx = (-1)^{mn}xy$  when  $x \in R_m$  and  $y \in R_n$ . Let  $\Lambda$  be an augmented  $R$ -algebra, and  $G$  a  $\Lambda$ -module. (We usually omit the word “graded”, but it is to be understood.) Unmarked tensor products are over the ground ring  $R$ .

We denote by  $(\Lambda|R)^\otimes$  the tensor algebra of  $\Lambda$  over  $R$ . Then  $(\Lambda|R)^\otimes \otimes_R G$  has a natural  $\Lambda\Gamma$ -module structure: if  $\varphi : [n] \rightarrow [m]$  is any morphism in  $\Gamma$ , we set

$$\varphi_*(\lambda_1 \otimes \cdots \otimes \lambda_n \otimes g) = \varepsilon \gamma_1 \otimes \cdots \otimes \gamma_m \otimes h$$

in which

$$\begin{aligned} \gamma_i &= \lambda_{i_1} \dots \lambda_{i_r} & \text{if } \varphi^{-1}(i) &= \{i_1, \dots, i_r\} \text{ where } i_1 < i_2 < \dots < i_r \\ h &= \lambda_{j_1} \dots \lambda_{j_s} g & \text{if } \varphi^{-1}(0) &= \{0, j_1, \dots, j_s\} \text{ where } j_1 < j_2 < \dots < j_s \end{aligned}$$

and in which  $\varepsilon$  is the sign of the permutation that rearranges  $\{1, 2, \dots, n\}$  in the order in which  $\lambda_1, \dots, \lambda_n$  appear in the expansion of the product  $\gamma_1 \dots \gamma_m$ . When  $\varphi$  is a permutation  $\sigma : [n] \rightarrow [n]$  this means that  $\varphi_*$  rearranges the factors and multiplies by the sign (compare [13, p.158])

$$\sigma_*(\lambda_1 \otimes \cdots \otimes \lambda_n \otimes g) = \varepsilon(\sigma) \lambda_{\sigma^{-1}1} \otimes \cdots \otimes \lambda_{\sigma^{-1}n} \otimes g.$$

**Definition 4.7.** The above  $\Lambda\Gamma$ -module  $(\Lambda|R)^\otimes \otimes_R G$  is called the *Loday functor*  $\mathcal{L}(\Lambda|R; G)$  since it was first defined in the ungraded case by Loday [11]. The functor  $\mathcal{L}(\Lambda|R; \Lambda)$  is also denoted  $\mathcal{L}(\Lambda|R)$  and is called the  *$\Gamma$ -cotangent complex* of  $\Lambda$  over  $R$ .

The  *$\Gamma$ -homology* and  *$\Gamma$ -cohomology* of  $\Lambda$  relative to  $R$ , with coefficients in the  $\Lambda$ -module  $G$ , are defined as the homotopy and cohomotopy of the Loday functor:

$$\begin{aligned} H\Gamma_*(\Lambda|R; G) &= \pi_*((\mathcal{L}(\Lambda|R) \otimes_\Lambda G)) \\ H\Gamma^*(\Lambda|R; G) &= \pi^* \text{Hom}_\Lambda(\mathcal{L}(\Lambda|R), G) \end{aligned}$$

Since the  $\Gamma$ -modules here are graded, all these constructs have a further internal grading.

By Pirashvili's theorem and Theorem 4.4 above, we can write  $\Gamma$ -homology as a Tor-group, and therefore as the homology of a  $\Xi$ -complex:

$$H\Gamma_*(\Lambda|R; G) \approx \text{Tor}_*^\Gamma(t, \mathcal{L}(\Lambda|R; G)) \approx H\Xi_*(\mathcal{L}(\Lambda|R; G)).$$

As the  $\Xi$ -complex is based upon a projective resolution of the right  $\Gamma$ -module  $t$ , this can be dualized to write  $\Gamma$ -cohomology in terms of Ext and the dual  $\Xi$ -cohomology complex:

$$H\Gamma^*(\Lambda|R; G) \approx \text{Ext}_\Gamma^*(t, \text{Hom}_\Lambda(\mathcal{L}(\Lambda|R), G)) \approx H\Xi^*(\mathcal{L}(\Lambda|R); G).$$

Our claim is that gamma homology of commutative algebras is a precise analogue of Hochschild homology of associative algebras; and the  $\Xi$ -complex of the Loday functor is the corresponding analogue of the standard Hochschild chain complex. As evidence for this, we show that gamma homology, like Hochschild homology, satisfies a normalization theorem.

**Proposition 4.8.** *The cross-effect functor  $\tilde{\mathcal{L}}(\Lambda|R; G)$  of  $\mathcal{L}(\Lambda|R; G)$  satisfies*

$$\tilde{\mathcal{L}}(\Lambda|R; G)(\underline{n}) = \text{cr } \mathcal{L}(\Lambda|R; G)(\underline{n}) = \tilde{\Lambda}^{\otimes n} \otimes G$$

where  $\tilde{\Lambda}$  is the quotient  $R$ -module  $\Lambda/R$ .

*Proof.* In our case  $\tilde{\Lambda}$  is isomorphic to the augmentation ideal, and the proposition is immediate from the construction of the cross-effect functor. The general non-augmented case is treated in [13, 1.10], where an explicit formula is given for the action of morphisms of  $\Omega$  on these tensor products.  $\square$

Now the complex  $\Xi(\Phi)$  depends only upon the action in  $\Phi$  of the surjections (face operators) of  $\Gamma$ . It can be defined for any  $\Omega$ -module  $\Phi$  by making surjections outside  $\Omega$  (that is, those in  $J$ ) act by zero.

**Corollary 4.9.** (*Normalization Theorem for  $\Gamma$ -homology*) *The  $\Xi$ -complex for this reduced Loday functor  $\tilde{\mathcal{L}}(\Lambda|R;G)$  also has homology  $H\Gamma_*(\Lambda|R;G)$ . The analogous result holds in cohomology.*

*Proof.* Since  $\tilde{\Lambda}$  is the augmentation ideal, we have  $\Lambda \approx \tilde{\Lambda} \oplus R$ , and the tensor power

$$\bigotimes^q \Lambda \approx \bigotimes^q (\tilde{\Lambda} \oplus R)$$

splits  $\Sigma_q$ -equivariantly as a direct sum  $\bigoplus_{i=0}^q V_i$ , where  $V_i$  is a sum of  $q!/i!(q-i)!$  copies of  $\bigotimes^i \tilde{\Lambda}$ . In the  $\Xi$ -complex of the Loday functor, the  $(q-1)$ st row  $\mathcal{B}(\text{Lie}_q^*, \Sigma_q, \bigotimes^q \Lambda \otimes G)$  splits accordingly. The homology of the row-summand containing  $V_i$  is zero if  $0 < i < q$  because  $V_i$  is induced up from a representation of  $\Sigma_i \times \Sigma_{q-i}$ , and  $\Sigma_i$  acts freely on  $\text{Lie}_q^*$  by 4.1(3). The row-summands with  $i = 0$  make up the  $\Xi$ -bicomplex for  $\mathcal{L}(R|R;G)$ , which is contractible. The summands with  $i = q$  make up the  $\Xi$ -complex of the reduced Loday functor; and this must be quasi-isomorphic to the whole of  $\Xi(\mathcal{L}(\Lambda|R;G))$  since we have shown that all the rest is contractible.

For the cohomology case, we use the Ext-interpretation of 4.7.  $\square$

In the next section, we shall further justify the analogy with Hochschild theory, by showing that the  $\Xi$ -complex arises in the commutative obstruction theory exactly as the Hochschild complex arose in the associative case.

## 5. OBSTRUCTION THEORY IN THE COMMUTATIVE CASE

**Resolving the commutative operad 5.1.** In the theory of ring spectra we used the Stasheff operad as a convenient resolution of the associative operad  $\mathcal{M}$ . In the commutative case  $\mathcal{M}$  is replaced by  $\mathcal{C}$ , where each space  $\mathcal{C}_n$  is a single point upon which the symmetric group  $\Sigma_n$  acts. To build our obstruction theory, we need a suitable resolution  $\mathcal{B} \rightarrow \mathcal{C}$ . It must satisfy two properties: each space  $\mathcal{B}_n$  should be contractible (in order for  $\mathcal{B}$  to be a resolution) and  $\Sigma_n$ -free (in order for  $\mathcal{B}$  to be cofibrant). That is,  $\mathcal{B}$  must be an  $E_\infty$  operad. However, being  $E_\infty$  is not sufficient. The Barratt-Eccles  $E_\infty$  operad  $\mathcal{D}$ , in which  $\mathcal{D}_n$  is the standard Eilenberg-Mac Lane model for  $E\Sigma_n$ , is not cofibrant because it fails the test that the faces (the images of compositions  $\mathcal{D}_i \times \mathcal{D}_j \rightarrow \mathcal{D}_n$ ) should intersect one another only in faces of faces. Better in this respect is the tree operad  $\mathcal{T}$ , (which differs from that

discussed in [18] only in that stumps are permitted). Thus  $\mathcal{T}_n$  is the  $\Sigma_n$ -space of trees having leaves labelled by  $\{1, 2, \dots, n\}$ , and stumps. Though  $\mathcal{T}_n$  is contractible and its faces intersect correctly, it is not  $\Sigma_n$ -free, and so  $\mathcal{T}$  is not an  $E_\infty$  operad.

**Definition 5.2.** Our *standard resolution* is the product operad  $\mathcal{B} = \mathcal{D} \times \mathcal{T}$ . Since its factors are augmented over the commutative operad  $\mathcal{C}$ , this is augmented over  $\mathcal{C} \times \mathcal{C} = \mathcal{C}$ ; furthermore, it inherits the facing properties of  $\mathcal{T}$  and the  $\Sigma$ -freeness of  $\mathcal{D}$ . It follows that  $\mathcal{B}$  is  $E_\infty$  and that  $\mathcal{B} \rightarrow \mathcal{C}$  is a cofibrant resolution of  $\mathcal{C}$ .

In analogy with 2.1 we have the following. The operads now have permutations, but the proof is otherwise exactly as before.

**Proposition 5.3.** *Let  $E$  be an  $S$ -module which is also a ring spectrum. Suppose that there is a morphism of operads  $\mathcal{B} \rightarrow \text{End}(E)$  under which one point of  $\mathcal{B}_2$  is mapped to the the given multiplication on  $E$ . Then  $E$  is weakly equivalent to a commutative  $S$ -algebra.*

**The geometry of the operad  $\mathcal{B}$  5.4.** The problem of replacing  $E$  by a commutative  $S$ -algebra is reduced by 5.3 to the problem of constructing a morphism of operads  $\mathcal{B} \rightarrow \text{End}(E)$ . This in turn we shall tackle by using obstruction theory. As before, we impose a strict unitality condition: we only look for actions of  $\mathcal{B}$  in which stumps are ignorable. In 3.7, this was equivalent to requiring the map of operads to commute with degeneracies. The operad  $\mathcal{B}$  inherits degeneracies from its factors  $\mathcal{D}$  and  $\mathcal{T}$ , and the ignorability of stumps (or the condition that  $\eta : S \rightarrow E$  be a strict unit) is interpreted just as before. It means that at the  $n$ th inductive step, the map is already defined on the large wedge subspectrum  $\bigvee^n E$  of  $E^{(n)}$ .

It is natural to try using induction on  $n$  to construct a sequence of  $\Sigma_n$ -equivariant maps  $\mathcal{B}_n \rightarrow \text{End}(E)_n = \text{Map}(E^{(n)}, E)$  which satisfy the conditions, as far as these are defined, for a morphism of operads. This would be a direct analogue of our procedure in §3, but it turns out to be too naive. It leads to intractable obstructions, and we need a better way.

We recall that  $\mathcal{B}_i = \mathcal{T}_i \times \mathcal{D}_i = \mathcal{T}_i \times E\Sigma_i$ . The bar construction  $E\Sigma_i$  has a well-known filtration: in the best-known model,  $E\Sigma_{ij}$  is the join of  $j + 1$  copies of the group  $\Sigma_i$ . Therefore  $\mathcal{B}_i$  is also filtered by setting

$$\mathcal{B}_{ij} = \mathcal{T}_i \times E\Sigma_{ij} ,$$

and the composition in the operad  $\mathcal{B}$  respects the filtration.

**Definition 5.5.** The *diagonal filtration*  $\nabla$  on the operad  $\mathcal{B}$  is defined using the bar filtration described above: we set  $\nabla^n \mathcal{B}_i = \mathcal{B}_{i, n-i}$ . An  $n$ -stage for an  $E_\infty$  structure on  $E$  is a family of  $\Sigma_i$ -equivariant maps  $\mu_i : \nabla^n \mathcal{B}_i \rightarrow \text{End}(E)_i$  preserving composition wherever defined, and such that a point of  $\nabla^n \mathcal{B}_2$  represents the given multiplication in  $E$ .

We remark that the 2-stage representing the given multiplication upon  $E$  can be extended to a 3-stage. In fact, this is exactly equivalent to stating that the multiplication is homotopy associative and homotopy commutative, by homotopies strictly preserving the unit.

**Extending an  $n$ -stage in the commutative case 5.6.** In the associative case, the problem of extending an  $\mathcal{A}_n$ -structure to an  $\mathcal{A}_{n+1}$ -structure leads one to consider the Stasheff polyhedron formed of all coherent bracketings of  $n + 1$  factors in fixed order. This polyhedron is a single cell of dimension  $n - 1$ .

The commutative case is more complex. First, the maps are required to be equivariant with respect to permutation of factors. Second, the lattice of coherent bracketings of  $n + 1$  ordered factors is replaced by the lattice of all partitions of the set  $\{1, 2, \dots, n + 1\}$ . The geometric realization of this lattice is a wedge of  $n!$  spheres of dimension  $n - 1$ , and the action of the group  $\Sigma_{n+1}$  upon its homology is the twisted dual  $\text{Lie}_{n+1}^*$  of the Lie representation [21] described in 4.1 above. The connection between  $E_\infty$ -structures and Lie representations, discovered by F. R. Cohen [7], underlies the appearance of the integral representations  $\text{Lie}_n^*$  in the bicomplex for  $\Gamma$ -homology. The next theorem gives the connection: it is a direct analogue of Theorem 3.9, with Hochschild cohomology replaced by  $\Gamma$ -cohomology. As before,  $R$  is the graded coefficient ring  $\pi_*E$ , and  $\Lambda$  the Hopf algebroid  $E_*E$ . A version of this theorem, with homotopy units in place of strict units, was published in [17].

**Theorem 5.7.** *Let  $E$  be a commutative ring spectrum which satisfies the perfect universal coefficient condition of 3.1. Then given an  $n$ -stage  $\mu$  for an  $E_\infty$  structure on  $E$ , there is a natural  $(n, 2 - n)$ -cocycle  $\theta(\mu)$  of the total complex  $\text{Tot } \Xi(\tilde{\mathcal{L}}(\Lambda|R; R))$  which vanishes if and only if there exists an  $(n+1)$ -stage extending  $\mu$ . The cohomology class  $[\theta(\mu)] \in H\Gamma^{n, 2-n}(\Lambda|R; R)$  is zero if and only if there exists an  $(n + 1)$ -stage which has the same underlying  $(n - 1)$ -stage as  $\mu$ .  $\square$*

**Corollary 5.8.** *If the groups  $H\Gamma^{n, 2-n}(\Lambda|R; R)$  are zero for all  $n \geq 3$ , then the commutative ring spectrum  $E$  has an  $E_\infty$  structure, and by 5.3 is therefore weakly equivalent to a commutative  $S$ -algebra.  $\square$*

The difference cochains belong to  $\Xi^{n, 1-n}(\tilde{\mathcal{L}}(\Lambda|R; R))$ . Therefore if the groups  $H\Gamma^{n, 1-n}(\Lambda|R; R)$  are zero for all  $n \geq 2$ , then  $E$  has at most one  $E_\infty$  structure. (The indexing of  $\Gamma$ -homology, like that of André-Quillen homology, differs by one from that of Hochschild homology, which is why the cohomological indices in 5.7 differ from those in 3.5)

The above results can be applied to a spectrum representing the Lubin-Tate theory corresponding to a Honda formal group law. Here  $H\Gamma^{**}(\Lambda|R; R) \approx 0$  [15], so 5.7 and 5.8 imply that these spectra have one and only one  $E_\infty$  structure. This reproves theorems of Goerss, Hopkins and Miller.

Baker and Richter [1] have further applications of the results. They prove that the Adams summand  $E(1)$  of the complex  $K$ -theory spectrum  $KU$  has one and only one  $E_\infty$  structure. Using a continuous  $\Gamma$ -cohomology, they prove that the completions of all the Johnson-Wilson spectra have unique  $E_\infty$  structures. (For the standard non-completed Johnson-Wilson spectra the question is still open.)

The analogue of Theorem 3.12 is also true in the  $E_\infty$  situation. The space of  $E_\infty$  structures on  $E$  (or of  $\nabla^n\mathcal{B}$ -structures for any  $n$ ) is unchanged up to weak homotopy type if one neglects the strict unit condition and relies upon the homotopy unit. The proof uses Corollary 4.9.

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