Notes for Juvitop, Spring 2017

Disclaimer: These are notes I took while attending the Juvitop student seminar at MIT. I have made them public in the hope that they might be useful to others, but these are not official notes in any way. In particular, mistakes are my fault; if you find any, please report them to:

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The TEX source for these notes can be found at
http://math.mit.edu/juvitop/notes_2017_Spring/juvitop-spring-2017-notes.zip.

1. February 15: Hood Chatham, Obstruction theory for A_∞ rings

This is based on a paper by Robinson of the same name.

Let R be an E_∞-ring, E an R-algebra (with potentially no associativity condition).

What does the A_∞-operad look like? (This is a non-symmetric operad.) In degrees 0, 1, and 2, you have a point; in degree 3 you have an interval; in degree 4 you have a pentagon; and in higher degrees you have higher associahedra. In general, $K_n \cong D^{n-2}$. Suppose we have an A_{n-1} structure on E, and we want an A_n-structure. We want an extension

$$\begin{tikzcd} \partial K_{n+} \wedge E^{\wedge n} \arrow[r] \arrow[d] & E \\ K_{n+} \wedge E^{\wedge n} \arrow[ur, dotted] & \end{tikzcd}$$

Recall $\partial K_n \simeq S^{n-3}$. This is just a "can I fill in a cell" obstruction problem, and the obstruction would live in $E^0(\partial K_{n+} \wedge E^{\wedge n}) \cong E^{3-n}(E^{\wedge n})$. *We're looking at the LES*

$$[\underbrace{\partial K_{n+}/K_{n+}}_{S^{n-2}} \wedge E^{\wedge n}, E] \to [\underbrace{K_{n+}}_{D^{n-2}} \wedge E^{\wedge n}, E] \to [\underbrace{\partial K_{n+}}_{S^{n-3}} \wedge E^{\wedge n}, E] \to [\underbrace{\Sigma^{-1}(\partial K_{n+}/K_{n+})}_{S^{n-3}} \wedge E^{\wedge n}, E].$$

Definition 1.1. Define $E_1^{s,t} := E^{-t}(E^{\wedge s})$.

Theorem 1.2. *The obstruction to extending A_{n-1} to A_n is some $c_n \in E^{n,n-3}$. If there is at least one extension, the set of them is an $E_1^{n,n-2}$-torsor. (Given one extension, look at maps on the cofiber $\Sigma\partial K_{n+} \wedge E^{\wedge n}$ of the vertical map above.)*

Here $\overline{E}$ is the cofiber of the unit map $R \to E$, and it appears because this is what happens when you require the map to respect the unit axiom.

Suppose E has an A_3-structure. Then E_1 is a cosimplicial group:

$$(d^i f)(x_1, \dots, x_{n+1}) = f(x_1, \dots, x_i x_{i+1}, \dots)$$

$$K_{n+} \wedge (E^{\wedge(i-1)} \wedge (K_{2+} \wedge E^{\wedge 2}) \wedge E^{\wedge(n-i)})$$

Notice that E_1 is not a spectrum; it is a bigraded group. Let E_2^{**} be the cohomology of E_1^{**}.

Theorem 1.3. *Given an A_{n-1}-structure for $n \geq 4$, the obstruction to extending the underlying A_{n-2} to A_n lives in $E_2^{n,n-3}$.*

If E is A_∞ and E_*E is projective over E_*[1], then there is a spectral sequence

$$\mathrm{Ext}_{\pi_*(E \wedge E^{op})}(E_*, E_*) \implies THC_R(E).$$

If E is not A_∞, then you can only define the differentials up to a certain stage: if E is A_{n-1}, where $n \geq 2r$, then the E_r page makes sense, and the obstruction to extending the A_{n-r}-structure to A_n lies there.

We're aiming towards the special case of Morava K-theories.

Morava K-theories are a regular quotient of Morava E-theory, which is an E_∞ ring spectrum.

Theorem 1.4. *If R is E_∞ and even (e.g. Morava E-theory), then every A_{n-1}-structure on R/I extends to an A_n-structure. By "extends", I mean that you might have to change the A_{n-1} structure, but not change the A_{n-2}-structure. (Here I is the ideal generated by a regular sequence.)*

Proof. We assumed that I was generated by a regular sequence; let that sequence be (g_i), where $|g_i| = 2d_i$. By a standard Koszul duality argument, $\pi_*(E \wedge_R E) = E_*(\alpha_i)$ where $|\alpha_i| = 2d_i + 1$. Then

$$\mathrm{Ext}_{\pi_*(E \wedge_R E^{op})}(E_*, E_*) = E_*[q_i]$$

where $|q_i| = (1, 2d_i + 2)$. So everything is in even total degree. Since $E_2^{n,n-3}$ is in odd total degree, it is zero. So there are no obstructions. (It might be a boundary, but you can change the A_{n-1} structure (without changing the A_{n-2}-structure) so this is actually zero.) $\square$

Applying this to $R =$ Morava E-theory, we see that any A_{n-1} structure on Morava K-theory extends in at least one way. The next question is how many A_∞ structures are there?

By the theorem, we're looking at $E_2^{j,j-2}$-torsors, and that's even so it's probably nonzero. Recall $|v_i| = 2(p^i - 1)$. So $\pi_*(K(n) \wedge K(n)^{op}) = K(n)_*[q_0, \ldots, q_{j-1}]$, where $|q_i| = 2p^i$. *Jun-Hou objects that these q_i's should be called t_i.* We want a monomial with j q_i's, and we want the topological degree to be 2. Suppose we have $q_{i_1} \ldots q_{i_j}$, and we can multiply by v_n^r (remember this r can be negative!). We want the degree of all of this to be 2. We have $|q_{i_1} \ldots q_{i_j}| = 2(p^{i_1} + \cdots + p^{i_j})$. The dimension (over $\mathbb{F}_p$) of $E_2^{j,j-2}$ is the number of such sequences $(i_1, \ldots, i_j)$ such that $p^{i_1} + \cdots + p^{i^j} \equiv 1 \pmod{p^n - 1}$. When $j < p$ there are zero ways (you can't get up to $p^n - 1$). If you have p of them, there are p choices, and for $j > p$ there are countably many choices (and so uncountably many A_∞ structures). There is possibly an issue with being able to go back and change things, but I claim this doesn't mess it up.

[1] Why this assumption? If E_*E is projective over E_*, then $\pi_* E^{\wedge(\bullet+1)}$ is a projective resolution of E_* over $\pi_*(E \wedge_R E^{op})$.

E_1-algebras and their modules. Let R be a ring [spectrum] and M, N left R-module[-spectra]. Then there are two spectral sequences

$$\begin{aligned}
\mathrm{Tor}^{\pi_* R}(\pi_* M, \pi_* N) &\implies \pi_*(M \wedge_R N)\\
\mathrm{Ext}_{\pi_* R}(\pi_* N, \pi_* M) &\implies \pi_* F_R(N, M)
\end{aligned}$$

Proof. Choose $F_0 \to N$ that is surjective on π_*. You can always arrange this, by $F_0 = \bigvee$ Let K_0 be the fiber of $F_0 \to N$ and let F_1 be a free R-module that surjects onto K_0 (i.e. on π_*). Then let K_1 be the fiber of $F_1 \to K_0$.

Get LES of each of the fiber sequences associated to

$$\begin{tikzcd}
N \arrow[r] & \Sigma K_0 \arrow[r] & \Sigma^2 K_1 \\
F_0 \arrow[u] & \Sigma F_1 \arrow[u] \arrow[l, dotted, "-1"] & \Sigma^2 F_2 \arrow[u] \arrow[l, dotted, "-2"]
\end{tikzcd}$$

The fact that the aforementioned maps are surjections in π_* says that the LES splits as $0 \to \pi_* K_i \to \pi_* F_i \to \pi_* K_{i-1} \to 0$. To get the spectral sequences, hit this diagram with the functors $\pi_*(M \wedge_R -)$ and $\pi_*(F_R(-, M))$. This gets

$$E^1_{s,t} = \pi_t(M \wedge_R F_s) = \pi_* M \otimes_{\pi_* R} \pi_* F_s.$$

This gives the Tor spectral sequence. You can get the Ext one similarly, with a projective resolution on the source. $\square$

If E is E_∞, then you get

$$\mathrm{Tor}^{E_* R}(E_* M, E_* N) \implies E_*(M \wedge_R N)$$

in the same way (the ring spectrum is then $E \wedge R$ and $E \wedge M$ and $E \wedge N$ are modules).

If E is homotopy commutative and $E_* R$ is flat over $\pi_* R$ is flat and even-graded, then $E_* M \cong E_* R \otimes_{\pi_* R} \pi_* M$. (There is a map in one direction $E_* M \leftarrow E_* R \otimes_{\pi_* R} \pi_* M$ by smashing with stuff, and in the other direction use the fact that it's a map of cohomology theories over R and it's an isomorphism when $M = R$.)

Let A be an E_1-ring of the form $A = R/(x_1, x_2, \dots)$ where $(x_1, \dots)$ is a regular sequence. Look at the spectral sequence

$$\mathrm{Tor}^{\pi_* R}(\pi_* A, \pi_* A^{op}) \implies \pi_*(A \wedge_R A^{op}).$$

Since you're quotienting by a regular sequence, you have a canonical choice of resolution, namely the Koszul resolution. Write $E = \pi_* R\langle \alpha_1, \dots \rangle$ where $|\alpha_i| = |x_i|$. Take the dg-algebra $\bigwedge^*_{\pi_* R} E$ such that $d\alpha_i = x_i$.

You get that the E_2 page of the spectral sequence is $\bigwedge_{\pi_* A} \langle \alpha_i \rangle$ where $|\alpha_i| = |x_i| + 1$.

For degree reasons all the differentials vanish. This works additively, but there might be multiplicative extensions – maybe $\alpha_i^2 \neq 0$. Note I never used the "op" (but then it would be false). You need to add another hypothesis, namely that A has a homotopy associative multiplication. *Maybe there is an issue at $p = 2$?*

Where is α_i^2? It must be hit by some element in $\pi_* A$. Since we have a homotopy-associative multiplication, A is an A-bimodule and we have a map $A \wedge A^{op} \xrightarrow{\varepsilon} A$ which induces an isomorphism on the 0-line of the spectral sequence. Homotopy associativity gives a diagram

$$\begin{tikzcd} A \wedge A^{op} \wedge A \wedge A^{op} \wedge A \arrow[r] \arrow[d] & A \wedge A^{op} \wedge A \arrow[d] \\ A \wedge A^{op} \wedge A \arrow[r] & A \end{tikzcd}$$

Chase the diagram for $(\alpha_i, \alpha_i, 1)$. One way around goes to 0 and one way goes to α_i^2. So under these hypotheses, $\alpha_i^2 = 0$.

Let R be an E_∞-algebra, A an E_1 R-algebra, and M an (A, A)-bimodule. This means that I can see it as an $A \wedge_R A^{op}$ left module, and an $A \wedge_R A^{op}$ right module (where secretly the second $A \wedge_R A^{op}$ is the "op" of the first). Then define

$$THH^R(A, M) = M \wedge_{A \wedge A^{op}} A$$
$$THH_R(A, M) = F_{A \wedge A^{op}}(A, M)$$

(So A is a left $A \wedge A^{op}$-module, and M is a right $A \wedge A^{op}$-module.) The previously discussed spectral sequences compute these things.

$THH_R(A, A)$ can be thought of as endomorphisms of A as an (A, A)-bimodule. So it's at least E_1. If A is discrete, this endomorphism ring is just the center. So you might hope for better commutativity in general.

Theorem 2.1 (Deligne conjecture)**.** *If C is a stable E_n-monoidal ∞-category (e.g. E_2 is a braided monoidal category), then* $\mathrm{End}(1)$ *has a canonical structure of an E_{n+1}-ring spectrum.*

"Proof". $E_1 \otimes E_n = E_{n+1}$. □

The following is the original statement of the Deligne conjecture.

Corollary 2.2. $THH_R(A) := THH_R(A, A)$ *is* E_2.

$THH_R(A)$ is sometimes thought of as the center, but that's unfair because it's not commutative. But you can iterate this until it becomes E_∞.

If C is the category of [dg] categories, $\mathrm{End}(1)$ is called the *Drinfeld center*.

We would like a more explicit construction of THH.

Claim 2.3.
$$THH^R(A, M) = \mathrm{colim}(M \leftleftarrows M \wedge A \,\substack{\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow}\, M \wedge A \wedge A \dots)$$

where the object you're taking the colimit is the cyclic bar construction and the $\wedge$'s are all $\wedge_R$'s. Think of arranging all the A's and the M in a circle; the d_i face map is just multiplication. In the homology (as opposed to cohomology) case:

$$THH_R(A, M) = \lim(M \rightrightarrows F(A, M) \rightrightarrows\!\!\!\!\to F(A^{\wedge 2}, M) \dots).$$

Proof. $THH^R(A, M) = M \wedge_{A \wedge A^{op}} THH^R(A, A \wedge A^{op})$. The claim is that the simplicial object $A \wedge A^{op} \rightrightarrows A \wedge A^{op} \wedge A \dots$ is just computing $A \wedge_A A = A$. $\square$

Why is this better? You can prove duality statements with it. For example, you can prove that THH^R and THH_R are dual. Unfortunately this does require some hypotheses. (In the general case I'm not sure why we're calling these "cohomology" and "homology".)

Proposition 2.4. *If M is a symmetric bimodule and A is E_∞, then*

$$THH_R(A, M) = F_A(THH^R(A), M).$$

Proof. Both sides are limits of the same cosimplicial diagram. $\square$

The real reason we care about this:

Theorem 2.5. *When A is E_∞, $THH^R(A)$ is also an E_∞-ring (it's a colimit of E_∞ rings and maps), and moreover $THH^R(A) = A^{\wedge_R S^1}$[2]. Furthermore, $THH^R(A) = N^{S^1}(A)$ where N is the HHR norm.*

Because it's the norm of an E_∞-spectrum, it automatically inherits the structure of a cyclotomic spectrum.

Let K be an operad in spaces. Let $\mathcal{O}$ be the category of totally ordered finite sets (this is sometimes called Δ^+ because it also contains $\emptyset$). Let $\mathcal{O}_*$ be the category of totally ordered finite sets with a distinct maximum and minimum. The objects are the same, but I also ask the maps to preserve the max and min. (This is sometimes called Δ^{op} because it's equivalent to Δ^{op} but this is a red herring.) Let $\mathcal{O}_K$ be the category with the same objects as $\mathcal{O}_*$ and $\mathrm{Map}(S, T) = \coprod_{f : S \to T \in \mathcal{O}_*} \prod_{t \in T} K_{f^{-1}t}$. (I've decorated every element of T with a way to multiply two objects in the preimage.)

Let A be an algebra over the operad K, and let M be a bimodule over the operad (this essentially means that $A \oplus M$ is an augmented algebra). Say A, M are in a category C.

Let K be the associative operad. *Actually, in Angeltveit's paper [next week's topic], he insists on using the associahedron model of the associative operad – there will be an underived smash product below, which is why this even has a chance of mattering.*

[2]I'm going to define $A^{\otimes K}$ for a simplicial set K: tensor levelwise with K (using disjoint unions) and then take geometric realization.

Theorem 2.6.
$$THH^R(A,M) = \int^{\mathcal{O}_K} W \wedge (S \mapsto M \wedge A^S)$$
$$THH_R(A,M) = \int_{\mathcal{O}_K} F(W, (S \mapsto F(A^S, M)))$$
where $\wedge$ is the underived smash product and W is a certain explicit object that depends on S.

Corollary 2.7. *A map $THH^R(A,M) \to B$ is the same thing as a natural transformation $W_S \wedge M \wedge A^S \to B$. A map $B \to THH_R(A,M)$ is the same thing as a natural transformation $B \wedge W_S \wedge A^S \to M$.*

If you replace K with $K_{\leq n}$, then you get $\mathrm{Tot}_{n-1}\, THH_R$ and $sk_{n-1}THH^R$ instead. You only need the A_n structure to define these skeleta/totalizations.

3. March 1: Andy Senger, "$THH_S(K_n) = E_n$ if you choose the right A_∞-structure on K_n"

Notation: K_n is the 2-periodic K-theory, not the v_n-periodic K-theory.

We know
$$(E_n)_* = \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]][u^\pm].$$
Define
$$K_n = E_n/(p_1, u_1, \ldots, u_{n-1}) = E_n/p \wedge_{E_n} E/u_1 \wedge_{E_n} \ldots \wedge_{E_n} E_n/u_{n-1}$$
(in order to define that smash product you need to use the fact that we have an E_∞ structure on E_n). I haven't yet fixed an A_∞-structure on K_n, so $THH_S(K_n)$ isn't defined (i.e. THH depends on the A_∞-structure).

There's a map $THH_{E_n}(K_n) \to THH_S(K_n)$ (this is THH-cohomology, not homology). Fact: this is an equivalence. (The motivation is that $S \to E_n$ is a K_n-local [pro]-Galois extension, so this has to do with Galois descent.)

I want to compute $THH_{E_n}(K_n)$. This is reasonable, because K_n is gotten by taking E_n and modding out by a regular sequence, and this sort of thing is easier to compute. We have a spectral sequence
$$\mathrm{Ext}^{**}_{\pi_*(K_n \wedge_{E_n} K_n^{op})}(K_{n*}, K_{n*}) \implies THH^*_{E_n}(K_n)$$
and moreover the LHS is actually computable: Denis proved in this sort of context that $\pi_*(K_n \wedge_{E_n} {K_n}^{op}) \cong \Lambda_{(K_n)_*}(\alpha_1, \ldots, \alpha_n)$ and $|\alpha_i| = |u_i| + 1 = 1$ where $u_0 = p$. But we know how to compute Ext over an exterior algebra! So our spectral sequence is
$$E_2 = (K_n)_*[q_1, \ldots, q_n] \implies THH^*_{E_n}(K_n)$$
where $|q_i| = (1, -1)$. This is great, because the q_i's are all in even total degree, and the spectral sequence collapses.

But this is not the end of the story: there could be nontrivial additive extensions (in fact there are lots of them) – the RHS is an $(E_n)_*$-algebra, and the u_i's can act nontrivially there.

The point of this talk is to understand the additive extensions in terms of the multiplication on K_n.

Now I'm going to look at a slightly more general context. Let R be an even E_∞-ring spectrum, $I = (x_1, \dots, x_n) \subset R_*$, and $A = R/I$. In this context the entire same thing is true: we still have a collapsing spectral sequence $A_*[q_1, \dots, q_n] \implies \pi_* THH_R(A)$ and we want to know about additive extensions. Since the RHS is an R_*-algebra, there is a map of rings $R_* \to \pi_* THH_R(A)$, and we just want to know where the elements $x_i \in R_*$ go. One way to approach this is to look at the following diagram:

$$\begin{tikzcd}
\Sigma^{|x_i|} R \arrow[r, "x_i"] \arrow[dr] & R \arrow[d, "\text{structure map}"] \\
& THH_R(A)
\end{tikzcd}$$

The diagonal map represents a homotopy element, and we want to know what it is. Continue the cofiber sequence

$$\begin{tikzcd}
\Sigma^{|x_i|} R \arrow[r, "x_i"] \arrow[dr] & R \arrow[r] \arrow[d] & R/x_i \arrow[dl, dotted] \\
& THH_R(A) &
\end{tikzcd}$$

The map $\Sigma R \to THH_R(A)$ (making the diagram commute) is the obstruction to getting the dotted map. You can think of this as lifting things in the Tot tower. Let's reduce to thinking about lifting one step in the Tot tower:

$$\begin{tikzcd}
\Sigma^{|x_i|} R \arrow[r, "x_i"] \arrow[dr] & R \arrow[r] \arrow[d] & R/x_i \arrow[dl, dotted] \\
& \mathrm{Tot}^1(THH_R(A)) &
\end{tikzcd}$$

This corresponds to finding the image of x_i modulo cohomological filtration of at least 2.

Let's write out the Tot tower:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & \mathrm{Tot}^1(THH_R(A)) \arrow[d] \\
R/x_i \arrow[r] \arrow[ur, dotted] & \mathrm{Tot}^0(THH_R(A)) = A
\end{tikzcd}$$

By using the definition and adjointing some things, this lifting problem is the same as the following: suppose we have a multiplication $R/x_i \wedge_R A \to A$. I'm interested in the map $(\partial A^1)_+ \wedge R/x_i \wedge_R A \to A$. (Here ∂A^1 is a 1-simplex.) There are two maps: φ (just the multiplication), and φ^{op} (which involves a swap map). The equivalent lifting problem is

$$\begin{tikzcd}
(\partial A^1)_+ \wedge R/x_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & A \\
(\Delta^1)_+ \wedge R/x_i \wedge_R A \arrow[ur, dotted] &
\end{tikzcd}$$

This is sort of a homotopy commutativity statement. We're working in spectra so we can just subtract the maps, and the obstruction is just $\varphi - \varphi^{op} : R/x_i \wedge_R A \to A$. But to be able to compute this, we need precise control over the A_2-structure on A.

In Hood's talk, we saw that any A_2-structure on A which extends to an A_3-structure extends to an A_∞-structure.

Let's do the simple case where $I = (x)$. I have $\varphi : R/x \wedge_R R/x \to R/x$, and smashing gives a map $\Sigma^{|x|} R/x \xrightarrow{x} R/x \to R/x \wedge_R R/x$. I claim that, because it's regular, this multiplication by x map is zero. There exists a section, and this is an A_2-structure. Suppose I have two different A_2-structures φ and $\overline{\varphi}$. We can view R/x as a 2-cell R-module, with the top cell attached by x. So this smash product is a 4-cell R-module. Unitality fixes what has to happen on the bottom 3 cells; they can only differ on the top cell. If I call projection to that cell $\beta : R/x \to \Sigma^{d+1} R$ where $d = |x|$ (because it's the Bockstein), the difference is $\varphi - \overline{\varphi} = u \circ (\beta \wedge \beta)$ where $u \in \pi_{2d+2}(R/x)$. Given any A_2-structure I can just add this map onto it, and the set of A_2-structures is a torsor for $\pi_{2d+2}(R/x)$ (I'm just saying that the difference factors through projection to the top cell).

I claim that all of them extend to A_3-structures. Associativity says that we can look at $\varphi \circ (\varphi \wedge 1 - 1 \wedge \varphi) = v \circ (\beta \wedge \beta \wedge \beta)$ (again unitality forces the only interesting thing to be on the top cell) for $v \in \pi_{3d+3}(R/x) = \pi_{3d+3}(R)/x$. But R was assumed to be an even spectrum, so $3d + 3$ is odd and $\pi_{3d+3}(R) = 0$, and our map is zero.

I want to compute $\varphi - \varphi^{op}$, but φ^{op} is another A_2-structure, so $\varphi - \varphi^{op} = c(\varphi) \circ (\beta \wedge \beta)$ for some $c(\varphi)$ (think of c as an invariant of φ). This tells us what $x \in \pi_* THH_R(R/x)$ is, modulo filtration ≥ 2.

What is the answer? It projects to something in filtration 1. If you work a bit you figure out that $x \equiv c(\varphi) q$ where this is the q of the THH spectral sequence (but there's just one because there's just one x). (The q corresponds to a β...) Sanity check: $x \in \pi_* THH_R(R/x)$ and there is a spectral sequence $(R/x)_*[q] \implies \pi_* THH_R(R/x)$. *Details in the paper were confusing... Hint:* $\pi_* F_R(R/x, R/x) = \Lambda_{(R/x)_*}(\alpha)$ *and* $0 = R/x \xrightarrow{\beta} R \to R/x \ldots$

The map $\Sigma^{|x_i|} R \to \mathrm{Tot}^1$ earlier represents an element of $\pi_{|x_i|}(\mathrm{Tot}^1)$, and that is the obstruction.

If $c(\varphi)$ is invertible, then
$$\pi_* THH_R(R/x) \cong R_{*}{}^{\wedge}_{(x)}.$$
(We're using that the spectral sequence converges strongly.)

In general there is a canonical map from the Bousfield localization $\varprojlim R/x^n = R^{\wedge}_{R/x} \xrightarrow{\cong} THH_R(R/x) = F(R/x, R/x)$.

I want to show that I can guarantee that there is such an A_2-structure on K_1 so that $c(\varphi)$ is invertible. Since the coefficient ring is a field, this is just saying it's nonzero.

If I choose $\overline{\varphi} = \varphi + u \circ (\beta \wedge \beta)$, then it's not hard to show that $c(\overline{\varphi}) = c(\varphi) + 2u$. Why? This is telling me something about $\overline{\varphi} - \overline{\varphi}^{op}$; one u is what I've added to the top cell; the other is from flipping the smash product in the top cell, and that introduces a sign. If 2 is invertible in R/x, and R is 2-periodic (so for example $\pi_{2d+2}(R/x)$ is not empty), this implies that we can choose $c(\varphi)$ to be pretty much anything we want – in particular, we can choose it to be invertible.

This immediately tells us that $E_1 \cong THH_{E_1}(K_1)$ away from 2 (and you can smash them together away from 2). At 2, you have to know that K_1 is not commutative; this is precisely saying that our thing isn't zero.

This was the case where I is generated by a single element. Now suppose $I = (x_1, \ldots, x_n)$.

(1) Given $\varphi_i : R/x_i \wedge_R R/x_i \to R/x_i$ I can form $\widetilde{\varphi} : R/I \wedge_R R/I \to R/I$, so there exists some sort of A_2-multiplication, and it extends to an A_3-structure.

(2) All A_2-structures on R/I that extend to A_3-structures (this is no longer an empty condition) can be written uniquely as $\widetilde{\varphi} \circ \prod_{i \neq j}(\mathbb{1} + v_{ij}\beta_i \wedge \beta_j)$ for some unique $\widetilde{\varphi}$ ("a diagonal one") (here $\prod$ means iterated composition). In theory I can smash more than two of these together, but then it no longer extends to an A_3-structure.

(3) Define an $n \times n$ matrix $C(\varphi) = (c_{ij}(\varphi))$ where
$$c_{ii}(\varphi) = c(\varphi_i)$$
$$c_{ij}(\varphi) = -v_{ij} - v_{ji}$$
where $\widetilde{\varphi} = \varphi_1 \wedge \ldots \wedge \varphi_n$.

(4) In particular, if R is 2-periodic and $n \geq 2$, then I can make $c(\varphi)$ invertible. ("Just futz around with stuff – you can choose the off-diagonal entries.") So the problem at $p = 2$ disappears once $n \geq 2$.

(5) Thinking about the maps $R/x_i \wedge_R R/I \to R/I$, you get
$$x_i \equiv \sum_j c_{ij}(\varphi) q_j \quad (\text{mod filtration} \geq 2).$$

(6) If I make $c(\varphi)$ invertible (which I can because everything is 2-periodic), then from the spectral sequence we find that $\pi_* THH_R(R/I) \cong R_I^\wedge$, which means the Bousfield localization $R^\wedge_{R/I} \to THH_R(R/I)$ is an isomorphism. ("This is some Weierstrass preparation thing in multiple variables.")

(7) This implies that there exists an A_∞-structure on K_n such that $E_n \to THH_{E_n}(K_n)$ is an isomorphism.

4. March 8: Morgan Opie, André-Quillen (co)homology and Beck modules

The goal is to set up an analogy for Jun-Hou's talk next week. Quillen describes this for a pretty general kind of category; I want to just gesture towards the larger theory.

Outline:

(1) Vague stuff about the general theory

(2) Specific case of the cotangent complex for rings (applying the general theory to R-algebras over a given R-algebra)

4.1. **General theory.** Given $R \to B \to A$ and an A-module, you get a derivation exact sequence

$$0 \to \mathrm{Der}_B(A, M) \to \mathrm{Der}_R(A, M) \to \mathrm{Der}_R(B, M) \to \dots.$$

Quillen wanted a framework that would extend this to a long exact sequence. He wants to work in an algebraic category $\mathcal{C}$ such that, given a particular object c, we can talk about the abelian objects in the slice category $\mathcal{C}/c$, called $(\mathcal{C}/c)_{ab}$, and such that the forgetful functor $(\mathcal{C}/c)_{ab} \to \mathcal{C}/c$ admits a left adjoint Ab (abelianization). We want to get a good model structure on these things. The philosophy is that we want to get a homology object $\mathbb{L}\,\mathrm{Ab}(X)$ for all $X \in \mathcal{C}$.

4.2. **Commutative R-algebras Alg_R.** What does abelianization look like? Because we have a trivial final object, $(\mathrm{Alg}_R)_{ab}$ is trivial. The solution is to make the final object nonzero, and work in Alg_R/A for some $A \in \mathrm{Alg}_R$. The strategy is to identify $(\mathrm{Alg}_R/A)_{ab}$ with something more concrete, and then compute the abelianization functor.

Proposition 4.1. *There is an equivalence of categories* $(\mathrm{Alg}_R/A)_{ab} \simeq \mathrm{Mod}_A$.

The goal is to compute $\mathrm{Alg}_R/A \overset{\mathrm{Ab}}{\to} (\mathrm{Alg}_R/A)_{ab} \overset{\simeq}{\to} \mathrm{Mod}_A$. Let's construct a map $\mathrm{Mod}_A \to (\mathrm{Alg}_R/A)_{ab}$: to a module M we associate $A \oplus M$ as underlying module, with multiplication given by $(a, y) \cdot (a', y') = (aa', ay' + a'y)$. The map to A is just projection. Call this object $A \rtimes M$. (This is the square zero extension of A by M.)

We need to show it's an abelian group object. To say it's an abelian object is to say that Homs into it have an abelian group structure. So we analyze maps into it.

Lemma 4.2. $\mathrm{Hom}_{\mathrm{Alg}_R/A}(B, A \rtimes M) \cong \mathrm{Der}_R(B, M)$

If we have a diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\varepsilon"'] & A \rtimes M \\ A & \end{tikzcd}$$

we can write $f = \varepsilon \oplus d_f$. The claim is that d_f is a derivation. The verification is not too tricky, because $f(bb') = (\varepsilon(bb'), d_f(bb'))$, and because it's an algebra morphism, this is

$$\begin{aligned} f(bb') &= (\varepsilon(b), d_f(b)) \cdot (\varepsilon(b'), d_f(b')) \\ &= (\varepsilon(b)\varepsilon(b'), \varepsilon(b)d_f(b') + \varepsilon(b')d_f(b)) \end{aligned}$$

Corollary 4.3. $A \rtimes M$ *is an abelian object of* Alg_R/A.

Proof. Exercise. □

Now we will determine the abelianization functor.

Definition 4.4. The augmentation ideal of $X \in \mathrm{Alg}_A/A$ is $\ker(X \xrightarrow{\varepsilon} A)$.

Definition 4.5. The *module of indecomposables* of a non-unital A-algebra is

$$Q_A(X) := \mathrm{coker}(X \otimes_A X \to X).$$

We care about this because we can define the module of Kähler differentials.

Definition 4.6. The B-module of Kähler differentials of an R-algebra B is defined to be

$$\Omega_R(B) = Q_B(I_B(B \otimes_R B)).$$

There is a universal derivation $B \to \Omega_R(B)$, and

$$\mathrm{Der}_R(B, M) \cong \mathrm{Mod}_B(\Omega_R(B), M).$$

There is an adjunction

$$A \otimes_{(-)} \Omega_R(-) : \mathrm{Alg}_R/A \leftrightarrows \mathrm{Mod}_A : A \rtimes (-).$$

This will pretty much solve all of our problems.

One can show that any abelian object can be written as $A \rtimes M$. So then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Alg}_R/A}(B, A \rtimes M) &\cong \mathrm{Der}_R(B, M) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_B}(\Omega_R(B), M) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_A}(A \otimes_R \Omega_R(B), M). \end{aligned}$$

André-Quillen homology is the left derived functor of abelianization. Let's compute it in this case.

For $A \in \mathrm{Alg}_R(A)$ and $M \in \mathrm{Mod}_A$, we define $AQ_*(A, R; M)$ as follows. Choose a resolution $P_\bullet \to A$ where P_n is a free commutative R-algebra on (a set of) generators X_n, where under degeneracies, $X_n \subset X_{n+1}$. Apply $A \otimes_{(-)} \Omega_R(-)$ levelwise. For $A \in \mathrm{Alg}_R/A$, define

$$\mathbb{L}_R(A) = A \otimes_{P_\bullet} \Omega_R(P_\bullet).$$

What does $\mathbb{L}$ stand for? Maybe "linearization". Then define

$$AQ_*(A, R; M) = H_n Ch(A \otimes_{P_\bullet} \Omega_R(P_\bullet) \otimes_A M)$$

(where Ch is the associated chain complex under the Dold-Kan correspondence). Note that:

$$AQ_0 \cong \Omega_R(A) \otimes M$$
$$AQ^0 \cong \mathrm{Der}_R(A, M)$$

Example 4.7. Look at $R \to R[t]$, it's easy to resolve this by R-algebras – you just take the constant thing and that's a cofibrant replacement; the maps are alternating 0 and 1. So you get that for any $A = R[t]$-module M,

$$AQ_*(R[t], R; M) = \begin{cases} M & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.8. *Here are some key properties:*

- *Given*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ A \arrow[r] & A' \end{tikzcd}$$

you get a map $A' \otimes_A \mathbb{L}_R(A) \to L_{R'}(A')$.

- *Given* $R \to B \to A$, *we get a distinguished triangle in* $D(\mathrm{Mod}_A)$*:*

$$\mathbb{L}_R(B) \otimes A \to \mathbb{L}_R(A) \to \mathbb{L}_B(A) \to \Sigma\mathbb{L}_R(B) \otimes_B A$$

which gives rise to a LES of André-Quillen homology groups

$$\cdots \to AQ_1(A,B;M) \to AQ_0(B,R;M) \to AQ_0(A,R;M) \to AQ_0(A,R;M) \to AQ_0(A,B;M) \to 0.$$

- *If* $R \to A$ *is surjective, then we have* $AQ(A,R;M) \cong \ker(R \to A)/\ker^2 \otimes_A M$.

Example 4.9. Let's compute $R \to R/(r)$ for r a non-zero divisor. (This is following Iyengar's notes; he describes a fairly explicit way to compute cofibrant replacements.) The strategy is that, in order to kill things, you want to attach polynomial generators. Suppose you have $A_\bullet$ which is supposed to be resolving A but fails at some point. Suppose $\alpha \in A_n$ that you want to be zero. Then add variables in degree ℓ (for $\ell \geq n+1$) corresponding to each surjection $[\ell] \twoheadrightarrow [n+1]$. This results in a simplicial object

$$R \leftleftarrows R[x_{01}] \Lleftarrow R[x_{001}, x_{011}] \Lleftarrow R[x_{0001}, x_{0011}, x_{0111}]$$

where $x_{01} \overset{d_0}{\mapsto} r$, $x_{01} \overset{d_1}{\mapsto} 0$, and in general $x_I \mapsto x_{I \cdot d_i}$ if it's surjective and zero otherwise.

Applying $R/(r) \otimes_{(-)} \Omega_R(-)$, you get

$$AQ_1(A,R;M) = M, \quad AQ_i(A,R;M) = 0.$$

Now use induction and apply the LES to

$$R \to \underbrace{R/(r_1,\ldots,r_{n-1})}_{R_{n-1}} \to \underbrace{R/(r_1,\ldots,r_n)}_{R_n}.$$

The inductive hypothesis is that $AQ_k(R_{n-1},R) = 0$ for $k \geq 2$ and $k = 0$, and using the previous computation, we get $AQ_k(R_n,R;M) = 0$ for $k \geq 2$, and we get a SES $0 \to M^{n-1} \to\ ?? \to M \to 0$. We want to show that it splits, but actually we don't have to do this because we can use $AQ(A,R;M) \cong \ker(R \to A)/\ker^2 \otimes_A M$.

5. March 15: Jun Hou Fung, Introduction to topological André-Quillen (co)homology

5.1. **Introduction to TAQ.** Last time, we had a commutative ring R and an R-algebra B. Morgan produced for us the cotangent complex $\Omega_{B/R}$. Given another R-algebra A, we have an adjunction

$$A \otimes_{(-)} \Omega_{(-)/R} : \mathrm{Alg}_R/A \rightleftarrows \mathrm{Mod}_A \simeq (\mathrm{Alg}_R(A))_{ab} : A \ltimes (-).$$

Given $M \in \mathrm{Mod}_A$, we defined André-Quillen homology

$$AQ^R_*(A,M) := \mathrm{Tor}^A_*(\Omega_{A/R}, M)$$

and similarly cohomology

$$AQ^*_R(A, M) = \mathrm{Ext}^*_A(\Omega_{A/R}, M).$$

Everything (including Ω) is derived; usually people write L instead of Ω for this object.

Today we'll do a topological analogue of this. Let R be an E_∞-ring, $B \in \mathrm{Alg}_R$, and $M \in \mathrm{Mod}_R$. Then we have functors

$$I_B : \mathrm{Alg}_B /B \to \mathrm{Alg}^{nu}_B$$
$$Q_B : \mathrm{Alg}^{nu}_B \to \mathrm{Mod}_B$$

(here $(-)^{nu}$ means non-unital) which can be defined using some diagrams:

$$\begin{tikzcd} I_B(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varepsilon"] \\ * \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} N \wedge_B N \arrow[r] \arrow[d] & * \arrow[d] \\ N \arrow[r] & Q_B(N) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

These are the analogues of the augmentation ideal and indecomposables, respectively.

Definition 5.1 (Basterra)**.** The cotangent complex is

$$\Omega_{B/R} := \mathbb{L}Q_B \mathbb{R}I_B(B \wedge^{\mathbb{L}}_R B) \in h\,\mathrm{Mod}_B$$

and topological André-Quillen cohomology and homology are

$$TAQ^*_R(B; M) := \mathrm{Ext}^*_B(\Omega_{B/R}, M) = \pi_{-*}F_B(\Omega_{B/R}, M)$$
$$TAQ^R_*(B; M) := \mathrm{Tor}^B_*(\Omega_{B/R}, M).$$

After this line I'll never write $\mathbb{L}$ etc. and just assume everything is derived.

Remark 5.2. We have a Quillen equivalence

$$K : \mathrm{Alg}^{nu}_B \rightleftarrows \mathrm{Alg}_B /B : I,$$

where K is square-zero extension. There is also a Quillen adjunction

$$Q : \mathrm{Alg}^{nu}_B \rightleftarrows \mathrm{Mod}_B : Z$$

where Z is the functor that forms the algebra with zero multiplication.

Proposition 5.3. $A \wedge_{(-)} \Omega_{(-)/R} : h\,\mathrm{Alg}_R /A \rightleftarrows h\,\mathrm{Mod}_A : A \ltimes (-)$

Corollary 5.4. $TAQ^k_R(A, M) \cong h\,\mathrm{Alg}_R /A(A, A \ltimes \Sigma^k M)$

Corollary 5.5. *There is a forgetful map* $TAQ^k_R(A, M) \to H^k(A; M)$.

Properties of TAQ (same as for AQ):

- functoriality
- transitivity (which gives a LES)
- flat base change (which gives additivity)

These look like the axioms for generalized homology theories. Indeed,

Theorem 5.6 (Basterra-Mandell)**.** *Every cohomology theory on* Alg_R/A *is equivalent to* TAQ *for some* A*-module* M*.*

Remark 5.7 (Stabilization)**.** This is not Kriz's original definition of TAQ. For $B \in \mathrm{Alg}_R/R$, you can construct the stabilization

$$\Sigma^\infty B := \mathrm{colim}_n\, \Omega^n(S^n \otimes IB)$$

Theorem 5.8 (Basterra-McCarthy, Basterra-Mandell)**.**

$$QN \simeq \mathrm{colim}\, \Omega^n(S^n \otimes N)$$

Corollary 5.9. $D_1(\mathbb{1}_{\mathrm{Alg}_R/R}) \simeq TAQ$ *(this means first Goodwillie derivative)*

Example 5.10. Let E be a connective spectrum. Then

$$TAQ^S(\Sigma^\infty_+\Omega^\infty E) \simeq E$$

5.2. **Postnikov towers of E_∞-rings.** Recall, if E is connective we have

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \tau_{\leq 2}E \arrow[r, "k_2"] \arrow[d] & \Sigma^4 H\pi_3 E \\
 & \tau_{\leq 1}E \arrow[r, "k_1"] \arrow[d] & \Sigma^3 H\pi_2 E \\
E \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leq 0}E \arrow[r, "k_0"] & \Sigma^2 H\pi_1 E
\end{tikzcd}$$

Remark 5.11. $[k_n] \in H^{n+2}(\tau_{\leq n}E, \pi_{n+1}E)$

Proposition 5.12 (Kriz, Basterra)**.** *if* R *is a commutative* E_∞*-ring, we have a Postnikov tower*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \tau_{\leq 2}R \arrow[r, "\widetilde{k_2}"] \arrow[d] & \tau_{\leq 2}R \ltimes \Sigma^4 H\pi_3 R \\
 & \tau_{\leq 1}R \arrow[r, "\widetilde{k_1}"] \arrow[d] & \tau_{\leq 1}R \ltimes \Sigma^3 H\pi_2 R \\
R \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leq 0}R \arrow[r, "\widetilde{k_0}"] & \tau_{\leq 0}R \ltimes \Sigma^2 H\pi_1 R
\end{tikzcd}$$

where the $\tau_{\le n}R$'s are constructed inductively using the pullback diagram

$$\begin{tikzcd}
\tau_{\le n+1}R \arrow[r] \arrow[d] & \tau_{\le n}R \arrow[d] \\
\tau_{\le n}R \arrow[r, "\widetilde{k}_n"] & \tau_{\le n}R \ltimes \Sigma^{n+2}H\pi_{n+1}R
\end{tikzcd}$$

in Alg_R. *(Here the $\ltimes$ thing is $\vee$ on spectra, but is a square-zero extension as a ring spectrum.)*

The proof is similar to the proof for connective spectra E, but relies on the following lemma.

Lemma 5.13. *Suppose $A \to B$ is an n-equivalence of connective E_∞-rings for $n \ge 1$. Then*

$$\pi_i\Omega_{B/A} = \begin{cases} 0 & i \le n \\ \pi_n A & i = n+1. \end{cases}$$

Proof of proposition, given lemma. We have $R \xrightarrow{f_0} H\pi_0 R =: \tau_{\le 0}R$. This can be made an E_∞-map. This is a 1-equivalence, so the lemma says that

$$\pi_i\Omega_{\tau_{\le 0}R/R} \cong \begin{cases} 0 & i = 0,1 \\ \pi_1 R & i = 2. \end{cases}$$

Get $c_0 : \Omega_{\tau_{\le 0}R/R} \to \Sigma^2 H\pi_1 R$. This corresponds to the k-invariant $\widetilde{k}_0 : \tau_{\le 0}R \to \tau_{\le 0}R \ltimes \Sigma^2 H\pi_1 R$. Then you can construct $\tau_{\le 1}R$ by pullback, and $R \to \tau_{\le 1}R$ is a 2-equivalent. Rinse and repeat... $\square$

Remark 5.14.

$$\begin{tikzcd}
{[\widetilde{k}_n]} \arrow[d, mapsto] & \in & TAQ^{n+2}(\tau_{\le n}R, H\pi_{n+1}R) \arrow[d] \\
{[k_n]} & \in & H^{n+2}(\tau_{\le n}R, \pi_{n+1}R)
\end{tikzcd}$$

Corollary 5.15. *A connective spectrum E has the structure of an E_4 ring iff all its k-invariants lift to TAQ k-invariants.*

5.3. Computation of $TAQ^*(H\mathbb{F}_p) := TAQ^*_S(H\mathbb{F}_p, H\mathbb{F}_p)$.

5.3.1. *Warm-up.* First let's try to show that $TAQ^1 \ne 0$. This uses the Postnikov thing in a way that's not circular. If you stare hard at the definition of stabilization, you find

$$TAQ^*(\mathbb{F}_p) \cong TAQ^{*+2}_{H\mathbb{F}_p}(\underbrace{S^2 \otimes H\mathbb{F}_p}_{Y}, H\mathbb{F}_p).$$

What is Y? Look at the auxiliary thing $X := S^1 \otimes H\mathbb{F}_p \cong THH^S(H\mathbb{F}_p, H\mathbb{F}_p)$. Bökstedt has shown that $\pi_* X = P(x)$ where $|x| = 2$. Then $Y := S^1 \otimes X \cong H\mathbb{F}_p \wedge_X H\mathbb{F}_p$. The Künneth spectral sequence gives $\pi_* Y = \Lambda(y)$ for $|y| = 3$.

Now consider the Postnikov tower but just as a ring. So it has two cells, one in degree zero and one in degree 2. So it's classified by a single k-invariant in $THH_S^{3+2}(\mathbb{F}_p, \mathbb{F}_p)$. By dualizing the Bökstedt calculation, we see that this is actually zero. So Y is a square-zero extension $H\mathbb{F}_p \ltimes \Sigma^3 H\mathbb{F}_p$. We have

$$TAQ^1(\mathbb{F}_p) \cong TAQ^3_{H\mathbb{F}_p}(H\mathbb{F}_p \ltimes \Sigma^3 H\mathbb{F}_p, H\mathbb{F}_p) \ni \mathbb{1}_{H\mathbb{F}_p \ltimes \Sigma^3 H\mathbb{F}_p}.$$

The goal is to show that $TAQ^*(H\mathbb{F}_p)$ is generated by this element in degree 1 under the Steenrod operations.

5.3.2. *Construction of the spectral sequence.* This part goes back to Haynes Miller and (the correct part of) the incorrect Kriz paper. When we're working over a field, $TAQ^*(\mathbb{F}_p) = \pi_*(QI(H\mathbb{F}_p \wedge H\mathbb{F}_p))^\vee$. Let $N = QI(H\mathbb{F}_p \wedge H\mathbb{F}_p)$. Let's construct a bar resolution of N in $\mathrm{Alg}^{nu}_{H\mathbb{F}_p}$ using the free nonunital commutative $H\mathbb{F}_p$ algebra monad in $H\mathbb{F}_p$-modules:

$$\mathbb{A} : X \mapsto \bigvee_{j>0} X^{\wedge_{H\mathbb{F}_p} j} / \Sigma_j.$$

We get

$$B_* N = \big(\mathbb{A}N \leftleftarrows\!\!\!\rightarrow \mathbb{A}^2 N \leftleftarrows\!\!\!\rightleftarrows \mathbb{A}^3 N \qquad \ldots \big)$$

You get

$$E_2^{s,t} = H_s(\pi_t Q B_* N) \implies \pi_{s+t}(QI(H\mathbb{F}_p \wedge H\mathbb{F}_p))^\vee.$$

What is this E_2-page? You want to simplify $\pi_* Q\mathbb{A}X$. Recall that π_* of a commutative $H\mathbb{F}_p$-algebra has an allowable action of the Dyer-Lashof algebra R. Let

$$F : \mathrm{Vect}^{gr} \rightleftarrows \mathrm{Alg}^{nu}_R : U$$

be the free-forgetful adjunction. Then $\mathbb{F} = FU$ is a comonad.

Lemma 5.16. $\pi_* \mathbb{A} X \cong \mathbb{F} \pi_* X$ *and* $\pi_* Q\mathbb{A}X \cong Q\mathbb{F}\pi_* X$

We have $\pi_* Q B_* N \cong Q(\mathbb{F}_*(\pi_* N))$ where $\pi_* N \cong IA$ (where A is the dual Steenrod algebra). Then

$$E_2^{s,t} = \mathcal{L}_s^{\mathbb{F}}(\mathbb{F}_p \otimes_R Q(-))(A)_t^\vee$$

where $\mathcal{L}_s^{\mathbb{F}}$ denotes the s^{th} comonad $\mathbb{F}$-left derived functor.

5.3.3. *Computing the E_2-page.* This E_2 page can be computed using a Grothendieck (composite derived functor) spectral sequence.

In general, if you have:

- functors $\mathcal{C} \xrightarrow{F} \mathcal{B} \xrightarrow{E} \mathcal{A}$
- a comonad $(\mathbb{T}, \varepsilon, \delta)$ on $\mathcal{C}$, and
- a comonad $(\mathbb{S}, \varepsilon', \delta')$ on $\mathcal{B}$

subject to some conditions[3], then you get a spectral sequence

$$E_{s,t}^{GSS,2} = \mathcal{L}_s^{\mathbb{S}} E(\mathcal{L}_t^{\mathbb{T}} F(c)) \implies \mathcal{L}_{s+t}^{\mathbb{T}}(EF)(c).$$

[3]Conditions for the Grothendieck spectral sequence: for every object $c \in \mathcal{C}$, you need $F\mathbb{T}c$ to be $\mathcal{L}_*^{\mathbb{S}}E$-acyclic and $E\mathbb{S}^{n+1}$ to be exact for all $n \geq 0$.

(See Basterra, "André-Quillen cohomology of commutative S-algebras", Proposition 7.1 for this version of the Grothendieck spectral sequence, and more generally for this lecture.)

In our case, the composite functor is $\mathbb{F}_p \otimes_R Q(-)$, and the spectral sequence is

$$E^{GSS,2}_{s,t} = \mathrm{Tor}^{\mathbb{D}}_s(\mathbb{F}_p, \mathcal{L}^{\mathbb{F}}_t Q(A)) \implies \mathcal{L}^{\mathbb{F}}_{s+t}(\mathbb{F}_p \otimes_R Q(-))(A)$$

where $\mathbb{D}$ is the comonad associated to the free-forgetful adjunction between the category of graded vector spaces and the category of unstable modules (i.e. modules over the Dyer-Lashof algebra with the unstable condition[4]). *This* Tor *is called UnTor in some places, e.g. Haynes Miller's paper referenced below.*

At $p = 2$, Miller ("A spectral sequence for the homology of an infinite delooping") showed:

Proposition 5.17 (Miller)**.** $E^{GSS,2}$ *is the homology of* $L(QA)$ *where*

- $L(QA) \cong \bigoplus_{n\geq 0} L(n) \otimes (QA)_n$
- $L(n) = \left\langle \mathrm{Sq}^I \,:\, I \text{ is admissible, ends with } \mathrm{Sq}^j, j > n+1 \right\rangle$

We also know the action by the Dyer-Lashof algebra

$$d_1(Sq^I \xi_i) = \mathrm{Sq}^{I,2^{i-1}-1}\, \xi_{i-1}$$

We can compute this. If $i > 1$ and $j > 2^i + 1$ then $\mathrm{Sq}^{I,j}\, \xi_i \mapsto \mathrm{Sq}^{I,j,2^{i-1}-1}\, \xi_{i-1}$. If $i > 1$ and $j = 2^i + 1$ then it receives a differential: $\mathrm{Sq}^I\, \xi_{i+1} \mapsto \mathrm{Sq}^{I,2^{i-1}}$. If $i = 1$, then you have $\mathrm{Sq}^I\, \xi_2 \mapsto \mathrm{Sq}^{I,3}\, \xi_1$. As a consequence, the E^2 page is

$$E^2 \cong \mathbb{F}_2 \left\langle \mathrm{Sq}^I\, \xi_1 \,:\, I \text{ is admissible and ends with } j > 3 \right\rangle.$$

The Grothendieck spectral sequence collapses for degree reasons, and the original bar spectral sequence computing TAQ also collapses. So the answer is

$$TAQ^*(\mathbb{F}_p) \cong \mathbb{F}_2 \left\langle \mathrm{Sq}^I\, \xi_1^\vee \,:\, I \text{ is admissible and ends with } j > 3 \right\rangle.$$

5.4. TAQ **from higher** THH**.** The image $\mathrm{Im}(TAQ^*(\mathbb{F}_q) \to [H\mathbb{F}_p, H\mathbb{F}_p]^* = A)$ consists of multiples of the Bockstein β. For example, $\mathrm{fiber}(H\mathbb{F}_2 \overset{Sq^4}{\to} \Sigma^4 H\mathbb{F}_2)$ cannot be an E_∞-ring.

TAQ is a colimit[5]

$$\Omega(S^1 \otimes H\mathbb{F}_p) \to \Omega^2(S^2 \otimes H\mathbb{F}_p) \to \cdots \to TAQ(\mathbb{F}_p).$$

We have $S^1 \otimes H\mathbb{F}_p = THH(\mathbb{F}_p)$, and we can call the next terms "higher THH", written $S^n \otimes H\mathbb{F}_p = THH^{[n]}(\mathbb{F}_p)$. That is,

$$TAQ_*(\mathbb{F}_p) = \mathrm{colim}_n\, THH^{[n]}_{n+*}(\mathbb{F}_p).$$

[4]At $p = 2$, the unstable condition is $Q^n x = x^p$ if $|x| = n$; at $p > 2$ the condition is $Q^n x = x^p$ if $2|x| = n$.

[5]*What is up with this $\otimes$? This is not just $\Sigma H\mathbb{F}_p$. The idea is that $\Sigma H\mathbb{F}_p$ is not an algebra, so instead you define $S^1 \otimes H\mathbb{F}_p$ by taking the pushout of*

$$\begin{tikzcd} H\mathbb{F}_p \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & ?? \end{tikzcd}$$

in algebras. If you do this to an E_n algebra, you get an E_{n+1}-algebra.

There exist Künneth-type spectral sequences that look like

$$E^2 = \mathrm{Tor}^{THH^{[n]}_*(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \implies THH^{[n+1]}_*(\mathbb{F}_p).$$

It is an unpublished preprint by Basterra-Mandell that this collapses for all n. What's actually been published is that it collapses for $n \leq 2p + 2$.

Basterra and Mandell, in their paper "Multiplication on BP", use this spectral sequence for BP to show that BP is E_4. The computation becomes too difficult to do at E_5, probably because some homology/homotopy groups fail to be even-concntrated, so the spectral sequences don't obviously work out. But given Tyler's recent result, something happens between 4 and 12, but I don't know whether this is reflected in the differentials of the spectral sequence or whether this is a separate issue.

6. March 22: Hood Chatham, BP is E_4

Last time we saw that, given a ring spectrum R, we have a "k-invariant" on the Postnikov tower that lives in TAQ:

$$\begin{tikzcd}
 & R[k] \arrow[d] & \\
R' \arrow[r, "f_{k-1}"'] \arrow[ur, dotted] & R[k-1] \arrow[r, "k"] & H\pi_0 R \vee \Sigma^{k+1} H\pi_k R
\end{tikzcd}$$

$R[k]$ indicates Postnikov section, and $k_n(f) \in TAQ^{k+1}_{E_n}(R; \pi_k R)$.

From now on, everything (including MU, etc.) will be p-localized.

Base case: we have an E_4 map $MU \to BP[0] = H\mathbb{Z}_{(p)} \to MU[0]$. (Because it's even, we could equally have said $BP[1]$.)

We're going to create an E_4 structure on BP under MU, and also create an E_4 section: we'll create $MU \xrightarrow{f} BP[2-1] \xrightarrow{g} MU[2k-1]$ that is E_4.

Spectra are E_0.

I have a spectrum-level k-invariant $k^0_{2k}(BP)$, and the question is whether there's a lift $k^4_{2k}(BP[2k-1]) \mapsto k^0_{2k}(BP)$. I definitely have $k^4_{2k}(MU)$ (because MU is already E_∞). This is the same as $k^4_{2k}(\mathbb{1}_{MU})$. I can also contemplate $f_* g^* k^4_{2k}(MU) \in TAQ^{2k+1}(MU, \pi_{2k} MU)$ (here I'm using the map $BP \to MU$ on the left and $MU \to BP$ on the right). This is in the right place to be a k-invariant for BP.

Aside: what do these pullbacks mean? You can read this off the following diagram:

$$\begin{tikzcd} & Y[k] \arrow[d] & \\ X \arrow[r, "f"] \arrow[dr, "k(f)"'] & Y[k-1] \arrow[r, "g_*"] \arrow[d] \arrow[dr] & Z[k+1] \arrow[d] \\ & \Sigma^{k+1} H\pi_k Y \arrow[r] & \Sigma^{k+1} H\pi_k Z \end{tikzcd}$$

(Maybe the f and g are swapped here...) The reason I need the section is so I can push forward and pull back my k-invariant.

By using $f_* g^* k^4_{2k}(MU)$ I get an E_4-structure on $BP[2k+1]$. In order to rebuild the next level of the inductive hypothesis, I need lifts

$$\begin{tikzcd} & BP[2k+1] \arrow[r, dotted] \arrow[d] & MU[2k+1] \arrow[d] \\ MU \arrow[r, "f"] \arrow[ur, dotted, "f_{2k+1}"] & BP[2k-1] \arrow[r, "g"] & MU[2k-1] \end{tikzcd}$$

The obstruction for the diagonal lift (getting f_{2k+1}) is $o_{2k} \in TAQ^*_{E_4}(MU; \mathbb{Z}_{(p)})$. I want to show that this group is even, so all the obstructions vanish. Then I need to prove that $TAQ^*_{E_4}(BP[2k+1]; \mathbb{Z}_{(p)})$ is even to get the other lift. In the paper they do these the same way; I'll do them differently just for fun.

$TAQ^*(MU; \mathbb{Z}_{(p)})$ is the space of E_4 ring maps $MU \to \underbrace{H\mathbb{Z}_{(p)}}_{=MU[0]} \vee \Sigma^* H\mathbb{Z}_{(p)}$ (this is by definition of TAQ). MU is a Thom spectrum; there is a canonical truncation map $MU \to H\mathbb{Z}_{(p)}$. Since it's nonempty, E_4-ring$(\Sigma^\infty_+ BU, H\mathbb{Z}_{(p)} \vee \Sigma^* H\mathbb{Z}_{(p)})$ is a torsor for this (with a chosen identification because we have a favorite map). This is

$$\begin{aligned} E_4\text{-space}(BU, SL_1(H\mathbb{Z}_{(p)} \vee \Sigma^* H\mathbb{Z}_{(p)})) &= \mathrm{Space}(B^4 BU, B^4 \Omega^\infty \Sigma^* H\mathbb{Z}_{(p)}) \\ &= \mathrm{Space}(B^4 BU, K(\mathbb{Z}_{(p)}, *+4)) \\ &= H^{*+4}(B^4 BU; \mathbb{Z}_{(p)}) \end{aligned}$$

By Bott periodicity, $B^4 BU = BU\langle 6 \rangle$, and we have a fiber sequence $K(\mathbb{Z}, 3) \to BU\langle 6 \rangle \to BSU$. You run the Serre spectral sequence, and it's polynomial on even generators.

So, the first lift exists, i.e. we have a f_{2k+1}.

Now we need the second lift: we need to compute that $TAQ^*_{E_4}(BP[2n+1]; \mathbb{Z}_{(p)})$ is even through degree $2n+1$. Sadly, BP is not a Thom spectrum, so we have to actually do work.

Warning: the paper heavily requires on the iterated THH paper, which is a lot of technical stuff required to make this all work.

We're going to iterate the cyclic bar construction. Jun Hou said (briefly) that $TAQ_{E_\infty} = D_1 \mathbb{1}_{E_\infty\text{-ring}}$. That is, $TAQ_{E_\infty} = \operatorname{colim}_{n\to\infty} \Omega^n(- \otimes (S^1)^n)$. To get TAQ_{E_n}, you stop at the n^{th} stage:

$$E \vee \Sigma^n TAQ_{E_n}(E) = E \otimes (S^1)^n = THH^{[n]}(E)$$

(this is THH (i.e the cyclic bar construction $\widetilde{B}_{cyc}E$) iterated n times, written $\widetilde{B}^n(E)$). There is a spectral sequence which Jun Hou (vaguely) mentioned last time:

$$\operatorname{Tor}^{\pi_* \widetilde{B}^j(E)}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \implies \pi_*(H\mathbb{Z}_{(p)} \vee \Sigma^{j+1} TAQ_{E_{j+1}})(E; H\mathbb{Z}_{(p)}).$$

The idea is we're going to run this four times.

Let's do this for $BP[2k+1]$ (in the range of dimensions where this looks like BP). The zeroth bar construction we know... $\pi_* BP[2k+1] = \mathbb{Z}[v_i]$ (this and subsequent equal signs are true in the appropriate range; you can sort of forget about the $[2k+1]$). Start with $\mathbb{F}_p$ instead of $\mathbb{Z}_{(p)}$, and use the Bockstein spectral sequence at the end. (Actually, we just need to show something is even, so we don't care about the Bockstein spectral sequence anyway.) Then

$$\operatorname{Tor}^{\pi_* BP}(\mathbb{F}_{(p)}, \mathbb{F}_{(p)}) = \Lambda(\sigma v_i)$$

There can't be any differentials or extensions because of "exterior algebra stuff" (probably degree reasons). This is the E_2 page, and the E_∞ page, and actually $\pi_*(\widetilde{B}_{cyc} BP[2k+1])$.

Now we do this again:

$$\operatorname{Tor}^{\Lambda(\sigma v_i)}(\mathbb{F}_{(p)}, \mathbb{F}_{(p)}) = \bigotimes \mathbb{F}_{(p)}[\gamma^{p^n} \sigma^2 v_i]/(-)^p$$

(the γ is a divided power). There are "clearly" extension problems. Use a comparison with MU, because MU has lots of power operations. There is a surjection from what's happening on MU onto this stuff, in the appropriate dimensions. I claim

$$Q^{p^{n+i}} \gamma^{p^n} \sigma^2 v_i = (\gamma^{p^n} \sigma^2 v_i)^p.$$

On the other hand,

$$Q^{p^{n+i}} \gamma^{p^n} \sigma^2 v_i = \gamma^{p^n} \sigma^2 (Q^{p^i} v_i) \equiv \gamma^{p^n} \sigma^2 v_{i+1} \quad \text{(mod decomposables)}.$$

(You have to spend actual effort showing that the Q's commute with σ and the differentials etc.)

Because of the comparison map, these extensions get resolved the same way for BP:

$$\pi_* \widetilde{B}^2_{cyc} BP[2n+1] = \mathbb{F}_p[\sigma^2 v_i]$$

in the appropriate dimensions. Do this again, and claim that there aren't any problems. Do it again; you get something even. Now do a Bockstein spectral sequence to get the $\mathbb{Z}_{(p)}$ thing, and this collapses because it's even. Now do a universal coefficient spectral sequence which collapses because it's all concentrated in dimension 0.

So I get the lift $BP[2k+1] \to MU[2k+1]$.

I have time, so I can go through the "Jeremy proof". The idea is to completely avoid the cyclic bar complex. There's a lot of technical content involved in setting up those Tor spectral sequences and getting the Dyer-Lashof operations to work. This argument avoids this by taking advantage of the fact that MU is a Thom spectrum.

This argument doesn't try to build an E_4 section. We compute $TAQ(MU)$, not $TAQ(BP)$; if you want a section, you'd have to compute $TAQ(BP)$.

We want to construct an E_4 map $\varphi : MU \to MU$ such that $\ker \varphi \supset \ker(MU \to BP)$, and $\mathrm{hocolim}_{n\to\infty}(\varphi^n) = X$. "By a Priddy-style argument"[6], $X \simeq BP$. It's an isomorphism on H^0. Since the attaching maps are nontrivial, the claim is that it has to carry up to be an equivalence.

So, I want an E_4 map $MU \to MU$. This is the same as a map $B^4BU \to SL_1MU$. I'm going to do obstruction theory to get this, so I'm looking for

$$\begin{tikzcd} & B^4SL_1MU[2n+1] \arrow[d] \\ B^4BU \arrow[r] \arrow[ur, dotted] & SL_1MU \end{tikzcd}$$

On the zero skeleton, it can be the map $MU \to H\mathbb{F}_p$ used to construct BP.

I have a counit $\Omega^4\Sigma^4BU \to BU$ (note BU is E_4). Delooping, I get

$$\begin{tikzcd} & & B^4SL_1MU[2n+1] \arrow[d] \\ \Sigma^4BU \arrow[r] \arrow[urr, "\text{“spectrum map”}"] & B^4BU \arrow[r] \arrow[ur, dotted] & SL_1MU \end{tikzcd}$$

Think of Σ^4BU as having the "free" E_4 structure. By "spectrum map", I mean that it's a space map, but after taking the Thom spectrum I get a map $MU \to MU[2n+1]$; but it's not a ring map, just a spectrum map.

In degree $2p^k - 2$, we don't care about controlling the lift – the lift exists. In all other degrees, I have a new generator, and we want to send it to zero. On the spectrum level, I can just pick a map that sends my new thing to zero. Basically, I need to argue that in these degrees, the class of maps $B^4BU \to B^4SL_1MU[2n-1]$ surjects onto the class of maps $\Sigma^4BU \to B^4SL_1MU[2n-1]$.

I have an AHSS for computing the B^4SL_1MU-cohomology of the BU stuff via cellular approximation. $B^4SL_1MU[2n-1]$ has $\mathbb{Z}_{(p)}$ in degree 6 (because of various degree shifts and SL_1 killing the bottom thing). I need to show that

$$H^*(B^4BU; \mathbb{Z}_{(p)}) \to H^*(\Sigma^4BU; \mathbb{Z}_{(p)})$$

is surjective when $* \neq 2p^n - 2$. The LHS has been computed to be $\mathbb{Z}_{(p)}[c_i]_{i+1\neq p^k} \otimes \mathbb{Z}_{(p)}[c_{p^k-1}]$ (where the c_i's are Chern classes), and the map sends the first c_i's to c_{i+2} and the other c_{p^k-1}'s go. . . somewhere else.

[6]There's a paper by Priddy called "cellular . . . BP" in which he gives a construction of BP. He starts with spheres for the bottom cell and attaches cells with nontrivial attaching maps to kill the right homotopy.

The AHSS is all even, so it collapses. But this is a map on the associated graded which is surjective. Then you want to show that the AHSS is convergent enough that this implies it's surjective on the level of spaces.

(This was all for extending in degrees $\neq 2p^k - 2$. For the $2p^k - 2$ case, you just have to find *any* lift, and you can do space-level obstruction theory to get one.)

You could try to do this with $B^6 BU$ instead of $B^4 BU$ and the issue is that the cohomology isn't even.

$$H^*(\underbrace{BU\langle 2n\rangle}_{B^{2n-2}BU}; \mathbb{F}_p) = H^*(BU)/(c_i : \sigma_p(i-1) < n-i) \otimes F(\beta P^1 \iota_{2n-3})$$

where σ_p is the digit sum when written in base p, F is the free unstable algebra, and ι_{2n-3} is the fundamental class in that degree. (There's a similar statement for odd truncation.) The point is that all the things without β's hit something; use the Wu formula.

7. April 5: Andy Senger, BP is not E_{12}

This is about Tyler Lawson's recent paper. Everything is at 2.

We have a composite $BP \to \tau_{\leq 0} BP = H\mathbb{Z}_{(2)} \to \mathbb{F}_2$. (This is essentially the original definition of BP.) It is well known what this does on homology: this induces a map $BP_* BP \to H_* H$ which is just the inclusion of the subalgebra $\mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] = \mathbb{F}_2[\xi_1, \xi_2, \dots]$.

Suppose BP is E_n. Then this composite is automatically E_n: the second map is automatically E_∞, and the first map (a truncation map) is automatically E_n. If you want to show that BP isn't E_n, you can look at the action of the Dyer-Lashof algebra. The Dyer-Lashof action on $H_* H$ was computed by Steinberger. First you can check whether $H_*(BP)$ is a subalgebra; if it wasn't, then we would have a contradiction. Unfortunately, $H_*(BP)$ *is* a subalgebra.

But we haven't used all the structure – this Dyer-Lashof structure only requires an H_∞ structure, but we (assume) we have an E_∞ structure. What does the additional coherence give? This allows me to define secondary operations. If you find one that starts in $H_*(BP)$ and doesn't land there, you win. But no one has every computed a secondary Dyer-Lashof operation before, or even defined one!

We need two things:

(1) a secondary operation that gives a contradiction if you can compute it;

(2) the ability to actually compute secondary Dyer-Lashof operations in $H_* H$.

I'm going to focus on (2). (1) is kind of tricky; Tyler tried a bunch of things and they didn't work – the Nishida relations kept causing it to not cause a contradiction. There's an obstruction theory, based on Goerss-Hopkins obstruction theory, to help find new operations. You can find some obstruction classes to BP being an E_∞ ring spectrum, and that gives some hints for what relations you need. I'm not going to write down the whole operation – that would take a whole board.

Let $\mathbb{P}_H^{E_n}(x, z_{30})$ denote the free E_n H-algebra (here $H = H\mathbb{F}_2$) on $S^2 \vee S^{30}$. (Here $|x| = 2$.) The claim is that, for $n \geq 12$, there is a relation

$$\begin{tikzcd}
\mathbb{P}_H^{E_n}(x, z_{30}) \arrow[r, "R"] \arrow[dr, "\varepsilon"'] & \mathbb{P}_H^{E_n}(x, y_5, y_7, y_9, y_{13}, y_8, y_{10}, y_{12}) \arrow[d, "Q"] & \\
 & \mathbb{P}_H^{E_n}(x) \arrow[r, "\xi_1^2"] & H \wedge H
\end{tikzcd}$$

Work in the category of E_n H-algebras under $P_H^{E_n}(x)$. This is going to be some secondary operation that takes in something of degree 2. Here ε is the unit in this category – roughly speaking, because it takes $x \mapsto x$ and everything else to 0.

Furthermore, $Q(\xi_1^2) = 0$ ($Q(-)$ means post-compose with the map $\mathbb{P}_H^{E_n}(x) \to H \wedge H$). (This is some complicated thing using the Steinberger relation.)

Now we're in the right context to define a secondary operation. This category is a topological category, so we can define a bracket $\langle \xi_1^2, Q, R \rangle$. The point of this talk is to justify how to attack the following claim:

Goal 7.1. $\langle \xi_1^2, Q, R \rangle \equiv \xi_5$ (mod decomposables).

Also we claim that the entire indeterminacy is contained in decomposables, but that ends up being easy for degree reasons.

Step 1: reduce to more reasonable functional operations. (This step depends heavily on what the relation is.) There's a map

$$\mathbb{P}_H^{E_n}(x, z_{14}) \xrightarrow{\overline{Q}} \mathbb{P}_H^{E_n}(x, y_4) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H$$

where $\overline{Q}(x) = x$, $\overline{z_{14}} = Q^{10}y_4 + x^2 Q^6 y_4$, $f(x) = b_1$, $f(y_4) = b_2$, $p(b_1) = \xi_1^2$ and $p(b) = 0$. There's a fact about how the Q's act on H_*MU that says $f\overline{Q}(z_{14}) = 0$. The second composition is also zero (or "zero", namely it sends things that aren't x to zero).

Now $\langle p, f, \overline{Q} \rangle$ is defined. Up to indecomposables,

$$\langle \xi_1^2, Q, R \rangle \equiv Q^{16}(\langle p, f, \overline{Q} \rangle).$$

This is essentially elementary, just using Adem relations.

Now I'm going to add on an extra map to the end and juggle once more:

$$\mathbb{P}_H^{E_n}(x, z_{14}) \xrightarrow{\overline{Q}} \mathbb{P}_H^{E_n}(x, y_4) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge_{MU} H$$

where the last map sends $\xi_1^2 \mapsto 0$. We have a diagram

$$\begin{tikzcd}
M \wedge MU \arrow[rr] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[ddrr] & & M \wedge MU \wedge H \arrow[d, shift left] \arrow[d] \arrow[d, shift right] \\
 & H \arrow[dr] & H \wedge H \arrow[d] \\
 & & H \wedge_{MU} H
\end{tikzcd}$$

where the RHS is the definition of $H \wedge_{MU} H$. There exists a juggling-type relation

$$i\left\langle p, f, \overline{Q}\right\rangle = \langle i, p, f\rangle\, \overline{Q}.$$

Now we want to compute the RHS.

We want to know $\pi_*(H \wedge_{MU} H)$ and we want to know what i does on homotopy. For the first one, there is a Künneth-type spectral sequence

$$\underbrace{\mathrm{Tor}^{MU_*}(\mathbb{F}_2, \mathbb{F}_2)}_{\Lambda(\sigma x_i)} \implies \pi_*(H \wedge_{MU} H)$$

(Here σ is to be read as "suspension".) This completely degenerates for degree reasons, and there are no extensions, so we get

$$\pi_*(H \wedge_{MU} H) \cong \Lambda(\sigma x_i).$$

To analyze i, we use a different Künneth spectral sequence

$$\mathrm{Tor}^{H_*MU}(\mathbb{F}_2, H_*H) \implies \pi_* H \wedge_{MU} H.$$

We need to understand the H_*MU-structure on H_*H. This is easy to write down: $H_*MU \cong \mathbb{F}_2[b_1, b_2, \dots]$ where $|b_i| = 2i$, and the map $H_*MU \to H_*H = \mathbb{F}_2[\xi_1, \xi_2, \dots]$ takes $b_n \mapsto 0$ if $n \neq 2^k - 1$, and $b_{2^k-1} = \xi_k^2$. The answer ends up being

$$\mathrm{Tor}^{H_*MU}(\mathbb{F}_2, H_*H) \cong \Lambda(\xi_i) \otimes \Lambda(\sigma b_n \,:\, n \neq 2^k - 1).$$

We were trying to figure out what i did to $\pi_*(H \wedge H)$; the point is that we know what i does on the E_2 page here. This doesn't automatically collapse, but using the first spectral sequence(?) it collapses for degree reasons. Modulo decomposables, we have

$$\sigma b_n \equiv \sigma x_n \text{ for } n \neq 2^k - 1$$
$$\xi_k \equiv \sigma x_{2^{k-1}-1}$$

When you have a map of ring spectra $S \to R$, in

$$\mathbb{P} \xrightarrow{y} R \wedge S \xrightarrow{p} R \wedge R \xrightarrow{i} R \wedge_S R$$

we have $\langle i, p, y\rangle = \sigma y$. So we get $\langle i, p, f\rangle = \sigma b_2$, and we just found out using the second spectral sequence that $\sigma b_2 \equiv \sigma x_2$ mod decomposables.

Then we have

$$Q^{10}(\sigma x_2) + Q^2Q^6(\sigma x_2) = Q^{10}(\sigma x_2) \equiv i(\left\langle p, f, \overline{Q}\right\rangle).$$

If $\left\langle p, f, \overline{Q}\right\rangle \equiv \xi_4$, then $i(-) \equiv \sigma x_7$. Actually, this is an iff.

The upshot is that it suffices to compute a single Q^{10}: we need

$$Q^{10}(\sigma x_2) \equiv \sigma x_7$$

in $\pi_*(H \wedge_{MU} H)$. The idea is to somehow realize $\sigma : \pi_k MU \to \pi_{k+1} H \wedge_{MU} H$ (for $k > 0$) as coming from a map of more structured objects. This is the map that takes $x_i \mapsto \sigma x_i$. The hope is to reduce some fact about how power operations act on σx_i to how they act on x_i.

The answer is a map $SL_1(MU) \to \Omega SL_1(H \wedge_{MU} H)$. How do we get this? Apply SL_1 to

$$\begin{tikzcd} MU \arrow[r] \arrow[d] & H \arrow[d] \\ H \arrow[r] & H \wedge_{MU} H \end{tikzcd}$$

to get

$$\begin{tikzcd} SL_1(MU) \arrow[r] \arrow[d] & SL_1(H) \simeq * \arrow[d] \\ * \simeq SL_1(H) \arrow[r] & SL_1(H \wedge_{MU} H) \end{tikzcd}$$

This is a homotopy coherent diagram of infinite loop spaces, so by definition of loops, this defines precisely the map I want.

This induces a map $\pi_M U \cong \pi_k SL_1(MU) \to \pi_{k+1} SL_1(H \wedge_{MU} H) \cong \pi_{k+1} H \wedge_{MU} H$. Then you show that this map is actually σ.

(Dennis: this really looks like Bökstedt's computation of THH.)

$SL_1(H \wedge_{MU} H)$ is an H E_∞-algebra but $SL_1(MU)$ is not, so you have to induce up. So I have a map $BSL_1(MU) \to SL_1(H \wedge_{MU} H) \hookrightarrow \Omega^\infty H \wedge_{MU} H$. Adjointing this over we get $\Sigma_+ BSL_1(MU) \to H \wedge_{MU} H$. Since the LHS is an H E_∞ ring spectrum we can get $H \wedge (BSL_1(MU))_+ \to H \wedge_{MU} H$. Now this is a map of E_∞ H-algebras.

This gives a map $H_*(SL_1(MU)) \to H_{*+1}(BSL_1(MU)) \to H_{*+1}(H \wedge_{MU} H)$. If I let $\langle - \rangle$ denote the Hurewicz image, then this sends $\langle x_n \rangle \mapsto \sigma x_n$. This map came from a map of E_∞ H-algebras, so this situation preserves the Dyer-Lashof action. This means that we just need to actually compute $Q^{10}(\langle x \rangle) \equiv \langle x_7 \rangle$ modulo $\ker(\sigma : H_*(SL_1(MU))) \to H_{*+1}(H \wedge_{MU} H)$.

I'll give an idea of how to approach this final reduction. We have a big Hopf ring $H_*(\underline{MU}_*)$ (where $\underline{MU}$ is the Ω-spectrum of MU). What we need to do now is compute the multiplicative Dyer-Lashof action in this, modulo $\ker(\sigma)$ (it turns out that this kernel is very large). $H_*(\underline{MU}_*)$ is described completely by Ravenel-Wilson. It turns out that there's a power operation $P_2 : MU^{2n} \to MU^{4n}(B\Sigma_2)$; this comes from the H^2_∞-structure on MU. It turns out that there's some sort of commutative diagram

$$\begin{tikzcd} MU^{2n} \arrow[r, "P_2"] \arrow[d, "\Lambda"] & MU^{4n}(B\Sigma_2) \arrow[d, "\Lambda"] \\ H_*(\underline{MU}_{2n}) \arrow[r, "\underline{Q}"] & H_*(\underline{MU}_{4n}) \widehat{\otimes} H^*(B\Sigma_2) \end{tikzcd}$$

that gives $\underline{Q}$ as some sort of total multiplicative Dyer-Lashof operation.

You can identify $MU^*(B\Sigma_2) \cong MU^*[[\alpha]]/[2]_F(\alpha)$. Now if you view P_2 as a map in these coordinates $MU^* \to MU^*[[\alpha]]/[2]_F(\alpha)$, it all comes down to the following calculation:

$$P_2(x_2) \equiv x_7 \alpha^3 \bmod (\alpha^4) \text{ and } MU\text{-decomposables.}$$

This ultimately lets you deduce $Q^{10}(\langle x \rangle) \equiv \langle x_7 \rangle \bmod \ker(\sigma)$.

(In the usual Hopf ring notation, where # is additive and $\circ$ is multiplicative, $\langle x_n \rangle = [1]\#([x_n] \circ b_1^{\circ n})$. Here b is essentially a suspension. The magic is that σ kills a ton of stuff: #-decomposables, $\circ$-decomposables, and the ideal $(b_2, b_3, \dots)$.)

8. April 12: Peter Haine, Introduction to Γ-homology

Here is the plan:

(1) Define functor homology
(2) Derive HH, HC, and Γ-homology as examples
(3) Application: E_∞-obstruction theory

The point is to get a conceptual framework that fits all of these things.

8.1. Functor homology.

For right now and most of the time, we'll have R be a discrete commutative ring. Write $\mathrm{Mod}(R)$ for the category of R-modules. (I'll use Mod_R for something else.)

Definition 8.1. Let C be a category.

- A left C-module is a functor $C \to \mathrm{Mod}(R)$.
- A right C-module is a functor $C^{op} \to \mathrm{Mod}(R)$.
- Write ${}_C\mathrm{Mod} := \mathrm{Fun}(C, \mathrm{Mod}(R))$ and $\mathrm{Mod}_C = \mathrm{Fun}(C^{op}, \mathrm{Mod}(R))$

Example 8.2. For $c \in C$ we have $R[C(-,c)]$ and $R[C(c,-)]$ (i.e. these take an object c' to the free R-module on the appropriate Hom-set). These are projective generators for ${}_C\mathrm{Mod}$ and Mod_C, respectively.

Suppose I have a left C-module and a right C-module. I want to produce an R-module.

Definition 8.3. If $F : C \to \mathrm{Mod}(R)$ and $G : C^{op} \to \mathrm{Mod}(R)$ are C-modules, then the *functor tensor product* is

$$G \otimes_C F := \int^{c \in C} G(c) \otimes_R F(c).$$

I can also analogously define functor Hom.

Remark 8.4. $R[C(-,c)] \otimes_C F \cong F(c)$.

Definition 8.5. $\mathrm{Tor}_n^C(G, F) := G \otimes_C^{\mathbb{L}^n} F$

We can derive Hochschild homology and cyclic homology from this general story.

8.2. *HH*, *HC*, *H*Γ.

Recall: the category of noncommutative sets $\mathbb{F}_{nc}$ has:

- objects: nonempty finite sets
- morphisms: $f : I \to J$ is a set map with a total order of $f^{-1}(j)$ for all $j \in J$

- composition: given $I \xrightarrow{f} J \xrightarrow{g} K$

$$(gf)^{-1}(k) = \mathop{\ast}_{j \in g^{-1}(k)} f^{-1}(j) = \coprod_{j \in g^{-1}(k)} f^{-1}(j).$$

  Here $\ast$ is the join of simplicial sets.

You can also do this with pointed sets: $\mathrm{Fin}_{*,nc}$.

Our goal is to get HH and HC from $\mathbb{F}_{nc}$-modules and $\mathrm{Fin}_{*,nc}$-modules.

Definition/Example 8.6. $B_R : \mathbb{F}_{nc}^{op} \to \mathrm{Mod}(R)$ is defined as the coequalizer of

$$R[\mathbb{F}_{nc}(-, \{0,1\})] \underset{1<0}{\overset{0<1}{\rightrightarrows}} R[\mathbb{F}_{nc}(-, *)]$$

Similarly, you can define $\overline{B}_R : \mathrm{Fin}_{*,nc}^{op} \to \mathrm{Mod}(R)$.

Definition 8.7. Let R be a ring, A an associative unital R-algebra, and M an A-bimodule. The *Loday functor* $\mathcal{L}(A, M) : \mathrm{Fin}_{*,nc} \to \mathrm{Mod}(R)$ sends $I_+ \mapsto M \otimes A^{\otimes I}$.

Note: $\mathcal{L}(A, A) : \mathbb{F}_{nc} \to \mathrm{Mod}(R)$.

Theorem 8.8 (Loday, Pirashvili-Richter)**.** *With the same notation as previously,*

$$HC_*(A) \cong \mathrm{Tor}_*^{\mathbb{F}_{nc}}(B_R, \mathcal{L}(A, A))$$
$$HH_*(A, M) \cong \mathrm{Tor}_*^{\mathrm{Fin}_{*,nc}}(\overline{B}_R, \mathcal{L}(A, M)).$$

Definition 8.9. Define a functor $L_R : \mathrm{Fin}_*^{op} \to \mathrm{Mod}(R)$ as the coequalizer of

$$R[\mathrm{Fin}_*(-, 1_+ \vee 1_+)] \underset{\chi_{1,*} + \chi_{2,*}}{\overset{\nabla_*}{\rightrightarrows}} R[\mathrm{Fin}_*(-, 1_+)] \cong \mathrm{Set}_*(-, (R, 0)).$$

Here ∇_* is the fold map and $\chi_{1,*}$ is the "characteristic map" that crushes the second factor.

(This is usually written as t.)

Definition 8.10. The Γ-homology of a functor $F : \mathrm{Fin}_* \to \mathrm{Mod}(R)$ is $H\Gamma_*(F) := \mathrm{Tor}_*^{\mathrm{Fin}_*}(L_R, F)$.

8.3. **Obstruction theory/ Robinson's view.** Robinson thinks about this stuff in a completely different way, that relates to E_∞ things. I'll talk about Γ-homology of the Loday functor, but there will be some modifications.

R is still a commutative ring.

Definition 8.11. Write Lie_n for the n^{th} module of the Lie operad. Equivalently, this is the submodule of the free Lie algbera on $x_1, \ldots, x_n$ spanned by the monomials with no repetitions. Equivalently, this is the module of natural transformations $U^{\otimes n} \implies U$ (here $U : \mathrm{Lie}(R) \to \mathrm{Mod}(R)$ is the forgetful functor).

A dual with eventually appear; this seems nice because the Lie operad and the commutative operad are Koszul dual.

Definition 8.12. Given $F : \mathrm{Fin}_* \to \mathrm{Mod}(R)$, construct a double complex $\Xi_{*,*}(F)$ as follows. First note that $F(n_+)$ has a natural Σ_n-action. Now we can do a 2-sided bar construction: the $(n-1)^{st}$ row of $\Xi_{*,*}(F)$ is

$$\mathrm{Bar}(\mathrm{Lie}_n^\vee, \Sigma_n, F(n_+)).$$

The action is permutation with sign. That is,

$$\Xi_{p,q}(F) = \mathrm{Lie}_{q+1}^\vee \otimes R[\Sigma_{q+1}^{\times p}] \otimes F((q+1)_+).$$

The horizontal differentials are bar differentials. The vertical differentials are complicated – look at the paper.

Definition 8.13. $H\Xi_*(F) = H_*(\mathrm{Tot}\, \Xi(F))$

Theorem 8.14. $H\Xi = H\Gamma$

You have a universal property for Tor, and you verify all the necessary things. You end up looking at a bunch of projective generators and do a computation that involves a ton of relations related to the vertical differentials.

Now work in the graded setting: R is a graded ring (think of this as the coefficient ring of a spectrum E), A is a commutative R-algebra (this will be E_*E), and M is a symmetric bimodule.

Definition 8.15. Define a twisted Loday functor $\mathcal{L}^\sigma(A, M)$ whose assignment on objects is the same as $\mathcal{L}(A, M)$, and on morphisms you introduce a sign. Then define

$$H\Gamma_*(A|R; M) = H\Gamma_*(\mathcal{L}^\sigma(A, M))$$
$$H\Gamma^*(A|R; M) = H\Gamma^*(\mathrm{Hom}_A(\mathcal{L}^\sigma(A, A), M)).$$

I'll leave you with a "why we care" theorem.

Theorem 8.16. *Start with an A_2-spectrum E so that the "dual Steenrod algebra" $\Lambda = E_*E$ is flat over $R = \pi_*E$. Also*

$$E^*(E^{\wedge n}) \cong \mathrm{Hom}_R(\Lambda^{\otimes n}, R).$$

Given an E_{n-1}-structure μ on E that can extend to an E_n-structure, a necessary and sufficient obstruction to extend μ to an E_{n+1}-structure lives in $H\Gamma^{n,2-n}(\Lambda|R; R)$.

If all of these vanish, uniqueness is related to $H\Gamma^{n,1-n}$.

Because this is a special case of functor homology, you get a bunch of structural spectral sequences. You can also realize this as the homotopy of some spectrum:

$$\begin{tikzcd} \mathrm{Fin}_* \arrow[r, "F"] \arrow[d, hook, "i"] & D(R) \\ \mathrm{Top}_*^{\mathrm{fin}} \arrow[ur, dotted, "i_! F"'] & \end{tikzcd}$$

(Here $D(R)$ is the derived category.) The spectrum is the first excisive approximation of the Kan extension $i_! F$.

9. April 19: Robin Elliott

Last time, we saw an introduction to $H\Gamma_*$ via the functor homology approach. We also saw briefly at the end that we can relate this to $\pi_*(\|F\|)$ where F is a Γ-module. The point was that you can use this to do E_∞-obstruction theory.

Next time, we will use this to get that there is a unique E_∞-structure on KU. The goal of this talk is to develop the properties of Γ-homology needed to show this result.

Peter alluded that Γ-homology is a shadow of something more general. Given a (sufficiently nice) operad, you can associate a homology theory on algebras of the operad. The intermediate step is you take an "operator category", and you produce functor homology on some functor associated to this. To show you that this is good for something, let's do some examples:

Ass	HH
Lie	Lie algebra (co)homology
Comm	Harrison homology ($\cong AQ$ in characteristic 0)
A_∞ operad	something like HH?
E_∞ operad	$H\Gamma$

We're working in chain complexes. HH has a cyclic cousin HC; Γ-homology also has a cyclic cousin $H\Gamma^{cy}$.

There's a cyclic/ non-cyclic duality which we're going to explore for a lot of this talk. Robinson thinks of a cyclic operad as like an operad, but where the output variable is on an equal footing as the input one. So you get a Σ_{r+1}-action on $\mathcal{C}(r)$.

I think it's possible to say all of this in the more general framework Peter introduced. There's also this issue that Robinson's paper is filled with off-by-one errors...

Definition 9.1. A (nonunital) cyclic operad $\mathcal{E}$ is a functor $\mathrm{Isom}(\mathrm{Fin}_{\geq 3}) \to \mathrm{Ch}(k\text{-Mod})$ with composition

$$\circ_{s,t} : \mathcal{E}_S \otimes \mathcal{E}_T \to \mathcal{E}_{S \sqcup_{s,t} T}$$

satisfying associativity and symmetry (i.e. equivariance w.r.t. the Σ-actions). (Here $S \sqcup_{s,t} T$ is a deleted sum, i.e. $(S\backslash s) \sqcup (T\backslash t)$.)

Remark 9.2. As before, there is a theory of cofibrant E_∞ cyclic operads.

Remark 9.3. We'll work with a specific cofibrant E_∞ cyclic operad $\mathcal{T}$, the tree operad. (This is the Borel construction on the contractible space of (unrooted) trees. Leaves are labeled by finite sets, and composition is by grafting of trees.)

Definition 9.4. Let $\mathcal{C}$ be a cyclic operad. A *cyclic $\mathcal{C}$-complex* is a functor

$$\mathcal{M} : \mathrm{Isom}(\mathrm{Fin}_{\geq 1})^{op} \to \mathrm{Ch}(k\text{-Mod})$$

(written $S \mapsto \mathcal{M}_S$) such that for each $\circ_{s,t} : \mathcal{C}_S \otimes \mathcal{C}_T \to \mathcal{C}_{S \sqcup_{s,t} T}$ we have a *formal adjoint*

$$\circ_{s,t}^* : \mathcal{C}_S \otimes \mathcal{M}_{S \sqcup_{s,t} T} \to \mathcal{M}_T$$

satisfying naturality and associativity conditions.

Example 9.5. For an algebra A over $\mathcal{C}$ with structure maps $\mu_V : \mathcal{C}_{V^0} \otimes A^{\otimes V} \to A$ (where V^0 is V adjoined a basepoint), take $\mathcal{M}_S = A^{\otimes S}$ and

$$\circ_{s,t}^* : \mathcal{C}_S \otimes A^{\otimes S \sqcup_{s,t} T} \cong \mathcal{C}_S \otimes A^{\otimes S\backslash s} \otimes A^{\otimes T\backslash t} \to A \otimes A^{\otimes T\backslash t} \cong A^{\otimes T}$$

that you should think of as partial multiplication.

Definition 9.6. A *non-cyclic $\mathcal{C}$-complex* $\mathcal{M}$ is a functor

$$\mathcal{M} : \mathrm{Isom}(\mathrm{Fin}_*)^{op} \to \mathrm{Ch}(k\text{-Mod})$$

such that for each

$$\circ_{0,1} : \mathcal{C}_{S^0} \otimes \mathcal{C}_{T^{01}} \to \mathcal{C}_{(S \sqcup T)^0}$$

you have

$$\circ_{0,1}^* : \mathcal{C}_{S^0} \otimes \mathcal{M}_{(S \sqcup T)^0} \to \mathcal{M}_{T^{0,1}}$$
$$\circ_{1,0}^* : \mathcal{C}_{T^{01}} \otimes \mathcal{M}^0_{(S \sqcup T)} \to \mathcal{M}_{S^0}$$

Example 9.7. If A is a k-algebra that is an algebra over the cyclic operad $\mathcal{C}$, and M is an A-module, we can do a similar construction as for the cyclic complex: let $\mathcal{M}_{S^0} = A^{\otimes S} \otimes M$ with $\circ_{0,1}^* = \mu_S \otimes 1$ (where μ is the algebra structure) and $\circ_{1,0}^* = 1 \otimes \nu_T$ (where ν is the module structure). Then $\mathcal{C}_{S^0}$ is the Γ-*cotangent complex*, denoted $\mathcal{K}$.

Example 9.8. Given a Γ-module F, regard $F(S^0)$ as the trivial chain complex. Then $\circ_{0,1}^* : F(S \sqcup T)^0 \to F(T^0)$ and $\circ_{1,0}^* : F(S \sqcup T)^0 \to F(S^0)$ are constant over $\mathcal{C}$.

Realizations exist in the cyclic and noncyclic $\mathcal{C}$-complex case, but we'll just focus on the noncyclic case. Let $\mathcal{C}$ be a cofibrant acyclic operad (think – the E_∞ operad we had at the start), and $\mathcal{M}$ a noncyclic $\mathcal{C}$-complex (think $\mathcal{K}$). The goal is to construct the realization $|\mathcal{M}|$. Then $H\Gamma_* = H_*(|\mathcal{M}|)$. This is a two-step process.

Step 1: Define

$$|\mathcal{M}|' := \bigoplus_{|V^0|\geq 3} \mathcal{C}_{V^0} \otimes \mathcal{M}_{V^0}/(\varphi_* x \otimes m \sim x \otimes \varphi^* m)$$

where $\varphi \in \mathrm{Mor}(\mathrm{Isom}(\mathrm{Fin}_*))$, also quotiented by $\circ_{01}(x \otimes y) \otimes m \sim \partial^{S,T}(x \otimes y \otimes m)$ where

$$\partial^{S,T} : \mathcal{C}_{S^0} \otimes \mathcal{C}_{T^{0,1}} \otimes \mathcal{M}_{(S\sqcup T)^0} \to \mathcal{C}_{S^0} \otimes \mathcal{M}_{S^0} \oplus \mathcal{C}_{T^{01}} \otimes \mathcal{M}_{T^{01}}$$

is given by $(1 \otimes \circ_{10}^*) \oplus (1 \otimes \circ_{01}^*)(\tau \otimes 1)$ where τ swaps factors.

Remark 9.9. $|\mathcal{M}|'$ has filtration given by $\bigoplus_{3\leq |V^0| \leq n}$. This gives rise to a spectral sequence.

Step 2: We have to fix things in low degrees "because of the stupid 3 thing":

$$|\mathcal{M}| = \mathrm{cofib}(|\mathcal{M}|' \xrightarrow{\varepsilon} \mathcal{M}_2 = \mathcal{M}_{\{0,1\}})$$

where $\varepsilon = \varepsilon_0 - \sum_{v\in V} \varepsilon_v$ where ε_v is from $\circ_{1,0}^*$ on the partition $V^0 = \{v\} \sqcup (V - \{v\})$ and ε_0 comes from $\circ_{0,1}^*$ on the partition $V^0 = V \sqcup \{0\}$.

You can do this for A_∞ as well as for E_∞.

Theorem 9.10. *When you do this, the homology of the A_∞-realization is the Hochschild homology.*

Theorem 9.11. *The aforementioned filtration gives rise to a spectral sequence*

$$E^1_{p-1,q} \cong H_q(E\Sigma_p \otimes_{\Sigma_p} (V_p \otimes \mathcal{M}_{p+1})) \implies H_{p+q-1}(|\mathcal{M}|)$$

where V_p is the representation of Σ_n on $H_*(T_p)$ (where T_p is one of the spaces in the aforementioned tree operad).

Setup: B is a strictly commutative algebra, flat over a commutative ring A., and M is a B-module. Write $\mathcal{K}(A; M)$ for the realization of the cotangent complex we saw earlier – $\mathcal{K}_{S^0} = A^{\otimes S} \otimes M$, and similarly for B. Define the cotangent complex

$$\mathcal{K}(A|B; M) = \mathcal{K}(B, M)/\mathcal{K}(A, M).$$

Fact 9.12. *In the strictly commutative case, the spectral sequence in the theorem simplifies to*

$$E^1_{p-1,q} \cong H_q(\Sigma_p; V_p \otimes B^{\otimes p} \otimes M) \implies H\Gamma_{p+q-1}(B|A; M).$$

Theorem 9.13. *We can identify $E^1_{p-1,0}$ as the Harrison homology* $\mathrm{Harr}_*(B|A; M)$*, defined below.*

Definition 9.14 (Shuffle product)**.** $(ab) \sqcup\!\sqcup (xy) = abxy \pm axby \pm axyb \pm xayb \pm xyab$

Definition 9.15. Take the complex that computes HH and quotient out by all nontrivial shuffles. Then Harr_* is the homology of this complex.

Fact 9.16. Harr$_*$ *agrees with* AQ_* *in characteristic zero. Also, so does the higher homology of* Σ_p. *Then in characteristic zero,*

$$H\Gamma_{p-1}(B|A;M) \cong AQ_*(B|A;M).$$

Theorem 9.17.

(1) *If* $B \supset A$ *are* A*-algebras such that* B *is flat over* A*, and* M *is a* $B \otimes_A C$*-module, then*
$$\mathcal{K}(B \otimes_A C|C;M) \cong \mathcal{K}(B|A;M)$$
is a quasi-isomorphism, and so you get an isomorphism in $H\Gamma_*$.

(2) *If* B *and* C *are flat* A*-algebras and* M *is a* $B \otimes_A C$*-module, then*
$$\mathcal{K}(B \otimes_A C|A;M) \cong \mathcal{K}(B|A;M) \oplus \mathcal{K}(C|A;M)$$
is a quasi-isomorphism so you get an isomorphism in $H\Gamma_*$.

(3) *If* B *is étale over* A*, then* $H\Gamma_*(B|A;M) \cong 0$ *for all* B*-modules* M.

10. May 3: Jeremy Hahn, Brauer group of Morava E-theory, Part 1

Fix a perfect field k of characteristic p and a formal group $\mathbb{G}_0$ of height n over k. Associated to this data is:

- A universal deformation $\mathbb{G}$ defined over $R \cong \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]]$
- A (Landweber-exact) cohomology theory E (Morava E-theory) with $\pi_* E \cong R[u^{\pm}]$ and $\operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \mathbb{G}$.

Example 10.1. Suppose $k = \mathbb{F}_p$ and $x +_{\mathbb{G}_0} y = x + y + xy$. Then $R \cong \mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$, $x +_{\mathbb{G}} y = x + y + xy$, and $E \cong KU_p^\wedge$.

Last semester, Danny gave a talk proving that the space of A_∞ structures on E is connected. Last lecture, Eva discussed that the space of E_∞ structures on E is connected – there is a essentially unique multiplicative structure. But to understand the full moduli space, you need more technical work of Goerss and Hopkins.

From the universal pair $(R, \mathbb{G})$, we can get a unique E_∞ ring spectrum E. What about over the original pair $(k, \mathbb{G}_0)$?

Definition 10.2. Morava K-theory $K(n)$ is the E-module $E/(p, u_1, u_2, \ldots, u_{n-1})$.

(Everything in the talk will be 2-periodic.)

This is just a module; in classical algebra, if you have a ring and you quotient by some elements, you expect to get a ring structure back. But last semester, we essentially proved:

Theorem 10.3. *There is no* E_∞*-ring structure on* $K(n)$.

Proof. Any E_2-ring with $p = 0$ (like Morava K-theory if it were E_∞) must be an $H\mathbb{F}_p$-algebra. So as just a spectrum, it is a wedge of (shifted) copies of $H\mathbb{F}_p$. But $H\mathbb{F}_p \wedge K(n) \simeq 0$ (because they're both field spectra so you can directly calculate it with a Postnikov tower). $\square$

We still might hope to get an analogue of the theorem Danny discussed, but this is what we've been talking about this semester, and it turns out that they're not essentially unique:

Theorem 10.4 (Robinson)**.** *There are uncountably many A_∞-structures on $K(n)$ in the category of E-modules.*

This is the theorem Hood told us about. But we also have:

Theorem 10.5 (Angeltveit)**.** *There is a unique A_∞-structure on $K(n)$ in the category of spectra.*

Theorem 10.6 (Angeltveit)**.** *There exists an A_∞ E-algebra structure on $K(n)$ which has E as its center.*

Goal 10.7 (Hopkins, Lurie, Hahn)**.** Understand the moduli of all of these *Azumaya* multiplications on Morava K-theory. In particular, we'll try to understand all Azumaya algebras, not just the ones on $K(n)$.

At odd primes you can have homotopy commutative multiplications, but they will never be Azumaya.

10.1. **Azumaya algebras.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category where $\otimes$ commutes with sifted colimits (in practice, it will commute with all colimits) (e.g. modules over some ring that has $\otimes$ commuting with colimits). Our primary example is where $\mathcal{C}$ is $K(n)$-local E-modules.

Definition 10.8. An A_∞-algebra A in $\mathcal{C}$ is Azumaya if the functor $X \mapsto A \otimes X$ induces an equivalence of $\mathcal{C}$ with the category of A-bimodules in $\mathcal{C}$.

(E.g. if $\mathcal{C}$ is just a 1-category, A_∞ here just means associative.)

Proposition 10.9. *If A is Azumaya, then the center of A is the unit $\mathbb{1}$.*

Proof. By definition, $\mathrm{Center}(A) \cong \mathrm{Hom}_{A\text{-bimod}}(A, A) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ and this is just $\mathbb{1}$ with its associative structure. $\square$

Definition 10.10. Two A_∞-algebras A_1 and A_2 in $\mathcal{C}$ are *Morita equivalent* if there is a $\mathcal{C}$-linear equivalence of categories

$$\mathrm{LMod}_{A_1}(\mathcal{C}) \simeq \mathrm{LMod}_{A_2}(\mathcal{C}).$$

(Here LMod means left modules.)

In practice, $\mathcal{C}$ is closed, but Jacob Lurie works in more generality.

Example 10.11. If k is a field, and $\mathcal{C}$ is the category of k-vector spaces, then the matrix algebra $\mathrm{End}(k^n)$ is Morita-equivalent to k itself.

Proposition 10.12. *Let A denote an A_∞-algebra in $\mathcal{C}$. Then the following are equivalent:*

(1) A is Azumaya.

(2) There exists an A_∞-algebra B such that $A \otimes B$ is Morita-equivalent to $\mathbb{1}$.

(3) All three of the following hold:
- *A is dualizable (think of this as a finiteness condition)*
- *A is full (this means that $- \otimes A$ detects weak equivalences)*
- *the natural map $A \otimes A^{op} \to \mathrm{End}(A)$ is an equivalence.*

The second condition in (3) is satisfied in $K(n)$-local E-modules for $A = K(n)$ sort of by definition, but is not true for E-modules.

Proof. We'll prove (1) $\iff$ (2). Let $\mathrm{Cat}_\infty^\sigma$ denote the ∞-category of ∞-categories that have sifted colimits, with sifted-colimit-preserving functors. This is symmetric monoidal under the cartesian product of categories. Then $\mathcal{C}$ as above is a commutative algebra object in $\mathrm{Cat}_\infty^\sigma$, and define $\mathrm{Mod}_\mathcal{C}^\sigma := \mathrm{Mod}_\mathcal{C}(\mathrm{Cat}_\infty^\sigma)$. This is what I mean by the category of $\mathcal{C}$-linear categories. If A is an A_∞-algebra in $\mathcal{C}$, then the category of A-bimodules is equivalent to $\mathrm{LMod}_A(\mathcal{C}) \otimes \mathrm{LMod}_{A^{op}}(\mathcal{C})$. There is always a functor $\mathcal{C} \to \mathrm{LMod}_A(\mathcal{C}) \otimes \mathrm{LMod}_{A^{op}}(\mathcal{C})$ given by $X \mapsto A \otimes X$, and the question is whether this is an equivalence. Since $\mathcal{C}$ is the unit in the category of modules over it, this presents $\mathrm{LMod}_A(\mathcal{C})$ as an invertible object, and by some abstract nonsense involving (2), this is an equivalence. (In this case $B = A^{op}$.) $\square$

Definition 10.13. The Brauer group of $\mathcal{C}$ has underlying set

$$\{\text{Azumaya algebras in } \mathcal{C}\} / \text{Morita equivalence}.$$

The group structure comes from tensor products of algebras.

Using (2), this gives a well-defined group structure.

If I'm working with the Brauer group, all I'm going to get is the Morava K-theories modulo Morita equivalence, not all the Morava K-theories. But it turns out to be OK, in the special case of $K(n)$:

Proposition 10.14. *Suppose $K(n)_1$ and $K(n)_2$ are two [Azumaya] Morava K-theories. If $K(n)_1$ is Morita equivalent to $K(n)_2$, then $K(n)_1$ is actually equivalent as an A_∞-algebra to $K(n)_2$.*

Proof. By assumption we have an equivalence of categories $\mathrm{Mod}_{K(n)_1}(E\text{-mod}) \cong \mathrm{Mod}_{K(n)_2}(E\text{-modules})$. Since $K(n)$ is a field spectrum, every module splits. Also this is 2-periodic, so every object in $\mathrm{Mod}_{K(n)_1}(E\text{-modules})$ is a wedge of $K(n)_1$'s and $\Sigma K(n)_1$'s. Coproducts go to coproducts under this equivalence of categories. So $K(n)_1$ goes either to $K(n)_2$ or $\Sigma K(n)_2$. But if I chose the equivalence that does the latter, I can just compose with the desuspension, so without loss of generality $K(n)_1$ gets sent to $K(n)_2$. Now the endomorphisms are the same. □

Aside: instead of modding out by Morita equivalence, you really want a Morita spectrum. Let $\mathrm{Cat}_\infty^\sigma$ be ∞-categories with sifted colimits under cartesian products. Consider $\mathcal{C}$, a commutative algebra object in $\mathrm{Cat}_\infty^\sigma$. Let $\mathcal{E} \subset \mathrm{Mod}_{\mathcal{C}}(\mathrm{Cat}_\infty^\sigma)$ be the full subcategories equivalent to $\mathrm{LMod}_A(\mathcal{C})$ where A is an Azumaya algebra in $\mathcal{C}$. *Actually you can remove the word "Azumaya".* Then let the "Brauer spectrum" $Br(\mathcal{C})$ be the E_∞-space of invertible objects in $\mathcal{E}$; this is the Picard group of the category $\mathcal{E}$, which is a connective spectrum. Then $\pi_0 Br(\mathcal{C})$ is the Brauer group of $\mathcal{C}$.

If R is an E_∞-ring in spectra, then $\tau_{\geq 0}\Sigma^{-2}Br(R\text{-mod}) \cong gl_1 R$. In particular, for Morava E-theory this implies the existence of an interesting map $Br(K(n) - \text{local } E\text{-modules}) \to E$; this is the Rezk logarithm. This has the same Bousfield-Kuhn functor and the same $K(n)$-localization.

Example 10.15. Suppose k is a field and $\mathcal{C}$ is the category of k-vector spaces. A k-algebra A is Azumaya if it is of the form $M_n(D)$ where D is a central division algebra (i.e. the center is just k) and M_n is a matrix algebra.

If you look at the Brauer group, you're only looking at these up to Morita equivalence, and that's just D itself – it doesn't see the different between D and matrix algebras over it.

Example 10.16. Let k be a field, and let $\mathcal{C}$ be the category of $\mathbb{Z}/2$-graded vector spaces ("super vector spaces") – an object looks like $V_0 \oplus V_1$ (even part $\oplus$ odd part), and the tensor product has the Koszul sign rule that swaps these around. There are Azumaya algebras that are not Azumaya when you forget about the grading. For example, suppose -1 is not a square in k. Then $k(\sqrt{-1}) \cong k \oplus k\sqrt{-1}$ is Azumaya (but is not Azumaya after forgetting the grading).

Let V be a vector space and $q : V \to k$ a nondegenerate quadratic form. Then the Clifford algebra Cl_q is the free algebra on V modulo the relation $x^2 = q(x)$ for all $x \in V$. It is $\mathbb{Z}/2$-graded with each $x \in V$ homogeneous of degree 1. Exercise: Check that this is a $\mathbb{Z}/2$-graded algebra which is Azumaya.

Definition 10.17. The Brauer-Wall group $BW(k)$ is the Brauer group of the category of $\mathbb{Z}/2$-graded k-vector spaces. For example, $BW(\mathbb{R}) \cong \mathbb{Z}/8$, generated by Clifford algebras. (A generator is one for the vector space $\mathcal{C}$.)

Back to the paper... we're interested in the case where $\mathcal{C}$ is the category of $K(n)$-local E-modules. We also have a functor from $\mathcal{C}$ to $\mathbb{Z}/2$-graded vector spaces over k taking $X \mapsto \pi_*(X \wedge_E K(n))$.

11. May 10: Jeremy Hahn, Brauer group of Morava E-theory, Part 2

Let $(\mathcal{C}, \otimes \mathbb{1})$ be a symmetric monoidal category where geometric realizations are preserved by $\otimes$.

Proposition 11.1. *Let A be an A_∞-algebra in $\mathcal{C}$. TFAE:*

1. *There exists an A_∞-algebra B such that $A \otimes B \sim \mathbb{1}$ (here $\sim$ means Morita equivalence).*
2. *The construction $X \mapsto A \otimes X$ yields a $\mathcal{C}$-linear equivalence of $\mathcal{C}$ with $\mathrm{Bimod}_A(\mathcal{C})$.*
3. *A is dualizable, full, and the natural map $A \otimes A^{op} \to \operatorname{End} A$ is an isomorphism.*

We call such an A *Azumaya*.

Proposition 11.2. *If $\mathcal{C}$ is presented with all colimits commuting with $\otimes$ then the center of A is the unit.*

Note from last time: it is not known whether the Brauer group is all the invertible things.

Example 11.3. Let k denote a field and $\mathcal{C}$ the category of $\mathbb{Z}/2$-graded k-vector spaces ("super vector spaces", i.e. with the Koszul sign rule). Then $Br(\mathcal{C}) = BW(k)$. If V is a k-vector space and $q : V \to k$ is a non-degenerate quadratic form, then the Clifford algebra is $\mathrm{FreeAlg}(V)/(x^2 = q(x))$. Then Cl_q is Azumaya. Moreover, $BW(\mathbb{R}) = \mathbb{Z}/8$ and you can get everything in it by these Clifford algebra constructions.

We're interested in the category of $K(n)$-local E-modules. Fix a perfect field k of odd characteristic and a formal group $\mathbb{G}$ of height n. We get a Morava E-theory which is an E_∞-ring in a unique way, and an E-module $K(n)$ with

$$\pi_* K(n) = \pi_* E/\mathfrak{m} = \mathbb{W}(k)[[u_1, u_2, \ldots, u_{n-1}]][u^{\pm}]/(p, u_1, \ldots, u_{n-1}).$$

I'll make one change from last time – at odd primes we can equip this with a homotopy commutative multiplication (but it won't be Azumaya). When I write $K(n)$ I'll be thinking of it with this multiplication (this is why I need the prime to be odd).

Let $\mathcal{C}$ be the category of $K(n)$-local E-modules. The construction $X \mapsto \pi_*(K(n) \wedge X)$ is a functor from $\mathcal{C}$ to $\mathbb{Z}/2$-graded k-vector spaces (the 2-fold periodicity on $K(n)$ gives rise to the grading). In order to make this functorial at the level of symmetric monoidal categories, you really need the homotopy-commutative structure on $K(n)$.

Whenever you have a functor of symmetric monoidal categories which sends full objects to full objects, you get a map of Brauer groups. In our case, you get a map $Br(\mathcal{C}) \to BW(k)$. (Recall the Brauer group Br is the group of Azumaya algebras under tensor product up to Morita equivalence.)

We can do a little better than just landing in $\mathbb{Z}/2$-graded k-vector spaces. X started life as an E-module, and $K(n) \wedge X$ really means $K(n) \wedge_E X$. That means there is an action of $\mathrm{End}_{E\text{-mod}}(K(n))$ on $\pi_*(K(n) \wedge X)$. This End is easy to calculate: it's $\Lambda[Q_0, Q_1, \ldots, Q_{n-1}]$

(exterior algebra) (the Q_i's kill the u_i's one by one). The notation is supposed to be reminiscent of the Milnor operators that kill elements in Postnikov towers.

Definition 11.4. A module over $\Lambda[Q_0, Q_1, \ldots, Q_{n-1}]$ is called a *Milnor module.*

There is a functor $F : \mathcal{C} \to$ Milnor modules which takes $X \mapsto \pi_*(X \wedge_E K(n))$. So we get a map $f : Br(\mathcal{C}) \to Br(\text{Milnor modules})$.

Theorem 11.5 (Hopkins-Lurie)**.** *The Brauer group of Milnor modules is $BW(k) \times$ the group of quadratic forms on $(\mathfrak{m}/\mathfrak{m}^2)^\vee$, where $\mathfrak{m}$ is the maximal ideal in E-theory.*

(This is a purely algebraic statement.)

Proposition 11.6. *An algebra in Milnor modules is a $\mathbb{Z}/2$-graded k-algebra A with a collection of odd derivations $\{d_v\}_{v\in(\mathfrak{m}/\mathfrak{m}^2)^\vee}$. The fact that the algebra is exterior means that $d_v^2 = 0$ and $d_{v+w} = d_v + d_w$. (Each Q_i is dual to a v.)*

A is Azumaya if A is Azumaya in $\mathbb{Z}/2$-graded algebras, and each derivation d_v is of the form

$$d_v(x) = a_v x + (-1)^{|x|} x a_v$$

for some scalar $a_v \in k$.

The association $v \mapsto a_v$ is a quadratic form on $(\mathfrak{m}/\mathfrak{m}^2)^\vee$.

Theorem 11.7. *$f : Br(\mathcal{C}) \to Br(Miln)$ is surjective (but not injective), and A is a Morava K-theory if $f(A)$ looks like (Cl_q, q).*

$X \mapsto \pi_*(K(n) \wedge_E X)$ is the same data as $\pi_0(K(n) \wedge_E X)$ and $\pi_0(\Sigma K(n) \wedge_E X)$. One of the key ideas is that $\pi_*(K(n) \wedge X) \cong \pi_* \operatorname{Hom}_{\substack{K(n)\text{-local} \\ E\text{-mod}}}(K(n), X)$. The point is that $K(n)$ is dualizable in the category of $K(n)$-local E-modules (it's self-dual). This is a somewhat cleaner perspective on what Goerss-Hopkins obstruction theory actually does.

Definition 11.8. The category Mol_E of *molecular E-modules* is the full subcategory of E-modules with objects = finite wedges of $K(n)$'s and $\Sigma K(n)$'s.

Here is the key idea of Jacob:

Proposition 11.9. *The category of Milnor modules is the category of functors $\mathrm{Mol}_E^{op} \to \mathrm{Set}$ that sends wedges to products.*

You're taking this category and freely adjoining all sifted colimits (if this were valued in spaces, this is called the nonabelian derived category).

Given this observation, it's natural to make the following definition:

Definition 11.10. A *synthetic E-module* is a functor $\mathrm{Mol}_E^{op} \to \text{Spaces}$ sending wedges to products.

There is a functor $\mathrm{Sy} : \mathcal{C} \to$ synthetic E-modules sending $X \mapsto \mathrm{Hom}(-, X)$. This immediately gives the following picture:

$$\begin{tikzcd}
 & \text{Synthetic } E\text{-modules} \arrow[d] \\
 & \vdots \\
 & \tau_{\leq 2}(\text{Synthetic } E\text{-modules}) \arrow[d] \\
 & \tau_{\leq 1}(\text{Synthetic } E\text{-modules}) \arrow[d] \\
\mathcal{C} \arrow[r, "F"] \arrow[ruuuu, "\mathrm{Sy}(-)"] & \text{Milnor modules} \simeq \tau_{\leq 0}(\text{Synthetic } E\text{-modules})
\end{tikzcd}$$

The idea is that you can get Azumaya algebras in synthetic E-modules by lifting Azumaya algebras through each of the smaller categories.

Proposition 11.11. Sy *is fully faithful.*

This is something to do with $K(n)$ being full. The idea is that this is some kind of Yoneda embedding. You don't know too much about its essential image, but you do know the following:

Proposition 11.12. *Every dualizable object in the category of synthetic E-modules is in the essential image.*

Every dualizable object in synthetic E-modules is in the image of a dualizable object of $\mathcal{C}$. Since any Azumaya object is dualizable, the Brauer space of $\mathcal{C}$ is equivalent to the Brauer space of synthetic E-modules.

What are the obstructions to lifting an Azumaya algebra (or A_∞-algebra)? It has to do with only the category of Milnor modules. If you stabilize synthetic E-modules, you get a T-structure, and Milnor modules is the heart. The obstructions to lifting are HH groups. They do this for every Azumaya algebra that happens to be a Morava K-theory and calculate the preimage of f.

Theorem 11.13. *The map $Br(\tau_{\leq n}\textit{Synthetic } E\textit{-modules}) \to Br(\tau_{n-1}\textit{Synthetic } E\textit{-modules})$ is surjective with kernel $(\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3})^\vee$.*

So now they know the Brauer group up to extension problems. Apparently this is still work in progress. There are two techniques: one is to use the Rezk logarithm; the other is an interesting way of actually constructing Azumaya K-theories. The computation shows that that you get all the Azumaya K-theories.

I'll discuss this construction. Consider $S^1 \to BGL(E)$; we have $\pi_1 BGL_1(E) = \pi_0 GL_1 E = (\pi_0 E)^\times$ and $S^1 \to BGL_1(E)$ is $1+u_i$ where u_i is one of the things in $\pi_* E$. Then $\mathrm{Thom}(1+u_i) = E/u_i$. If you want to build K-theory, take the map $\underbrace{S^1 \times \ldots \times S^1}_{n} \to BGL_1(E)$ by any regular sequence that kills the maximal ideal of E (e.g. $(1+p, 1+u_1, \ldots, 1+u_{n-1})$). Take the Thom spectrum of the product (which smashes together all the individual Thom spectra), and that's Morava K-theory as an E-module. Say you wanted to build this as an Azumaya A_∞-algebra. ll you have to do is check this has the structure of a loop map. So you want to build a map $\underbrace{\mathbb{C}\mathrm{P}^\infty \times \ldots \times CP^\infty}_{n} \to B^2 GL_1(E)$ such that when you loop this you get $S^1 \times \ldots \times S^1 \to BGL_1(E)$. So the question is how many of these are there? We're in the situation where classical obstruction theory works, and this is easy. They have various ways to check whether the result is Azumaya, homotopy-commutative, etc. It's easy to compute whether things have the right centers, because it's easy to compute THH on these things.

12. May 17: Allan Yuan, E_∞ rings from displays

Theorem 12.1 (Lawson)**.** *Let $h \geq 2$. There is an E_∞ ring spectrum E such that*

(1) $E_* = (\mathbb{Z}[u_1, \ldots, u_{h-1}])^{\widehat{}}_{(p,u_1)}[u^\pm]$

(2) the formal group of E extends the Lubin Tate formal group.

In particular, there is a map from E to the Lubin Tate spectrum, and tensoring along that map produces the Lubin Tate formal group.

Remark 12.2.

$$BP \to BP\langle n\rangle \to E(n) \to E_n$$

The input is that E_n is E_∞. The content of this theorem is that if you complete a little less, it's still E_∞.

Our goal today is to sketch a proof of this, given the p-divisible groups theorem.

Theorem 12.3 (Lurie, not written up)**.** *Let $\mathcal{N}$ be a Deligne-Mumford stack which is formal over $\mathbb{Z}_p$ (p is nilpotent and stuff is complete). Suppose we are given $\mathbb{G} : \mathcal{N} \to \mathcal{M}_p(h)$ (where $\mathcal{M}_p(h)$ is the moduli of p-divisible groups of height h) that is formally étale. Then there exists a sheaf $\mathcal{E}$ of E_∞-rings on $\mathcal{N}$ such that:*

(1) $\pi_0 \mathcal{E} = \mathcal{O}_\mathcal{N}$

(2) It's weakly even periodic.

(3) The formal group of $\mathcal{E}$ is $\mathcal{G}^{for}$ (the formal part).

Think of formally smooth as locally like $k \to k[[x_1, x_2, \ldots]]$, formally étale as like $k[[x_1, \ldots, x_t]] \to k[[y_1, \ldots, y_t]]$ (just have to check this on tangent spaces).

Remark 12.4. $\mathcal{N}$ doesn't have to be affine, so instead of getting one E_∞-ring you're getting a sheaf of them. The condition is local. Given a p-divisible group $\mathbb{G}$, over an algebraically closed field of positive characteristic there is a SES

$$0 \to \mathbb{G}^{\text{for}} \to \mathbb{G} \to \mathbb{G}^{\text{ét}} \to 0$$

If you believe Lubin-Tate theory you know about the deformation theory on $\mathbb{G}^{\text{for}}$: it looks like $W(k)[[u_1, \dots, u_{n-1}]]$. Also, $\mathbb{G}^{\text{ét}}$ looks like $(\mathbb{Q}_p/\mathbb{Z}_p)^{h-n}$. So you just have to put these things together. (Here h is the height of $\mathbb{G}$ and n is the formal height.) The universal deformation of $\mathbb{G}$ lives over $\mathbb{W}(k)[[u_1, \dots, u_{n-1}, t_1, \dots, t_{h-n}]]$.

So if you have something formally smooth, understanding the deformation theory reduces to understanding it on tangent spaces.

The plan is to give a simple algebraic approximation $\mathcal{M}_D$ to $\mathcal{M}_p(h)$ such that:

(1) $\mathcal{M}_D \sim \mathcal{M}_p(h)$ locally

(2) $\operatorname{Spf} R \to \mathcal{M}_D$ should have computable deformation theory.

The first candidate for this is a Dieudonné module.

Definition 12.5. Let k be a field of characteristic $p > 0$. Let f, v denote the Frobenius and Verschiebung, respectively, on $\mathbb{W}(k)$. Let the Dieudonné module be:

$$D_k = \mathbb{W}(k)[F, V]/\big(FV = VF = p, F(ax) = f(a)F(x), aV(x) = V(f(a)x)\big).$$

Theorem 12.6. *Let k be a perfect field. Then there is an equivalence*

$$\{p\text{-divisible groups over } k\}^{op} \xrightarrow{\sim} \{\text{modules over } D_k \text{ that are free and finite over } \mathbb{W}(k)\}.$$

The issue is the restriction that k be perfect. Displays will be a generalization of Dieudonné modules that works over non-perfect fields.

Remark 12.7.

- $\mathbb{W}(k)/p\mathbb{W}(k) \simeq k$. This is not true in general; in general, $\mathbb{W}(R)/V\mathbb{W}(R) = R$.
- The Dieudonné module also sees the tangent space: $DM(\mathbb{G})/V\,DM(\mathbb{G}) \simeq \operatorname{Lie}(\mathbb{G})$.
- There are a lot of versions of this correspondence. It originally comes from a correspondence

  $$\{\text{finite flat group schemes}\}^{op} \xleftarrow{\sim} \{\text{finite } \mathbb{W}(k) \text{ length } D_k\text{-modules}\}$$

  which comes from

  $$\{\text{formal groups}\}^{op} \hookleftarrow \{D_k\text{-modules, } V \text{ nilpotent}\}$$
- $\mathbb{G}(M) : A \mapsto \widehat{W}(A) \otimes_{W(k)} M/(F, V \text{ actions})$

Let R be a ring (or formal $\mathbb{Z}_p$-algebra).

Definition 12.8. A display over R is a tuple (P, Q, F, V^{-1}) where

- P is a finitely generated locally free $W(R)$-module

- $I_R P \subset Q \subset P$ (where Q is to be thought of as the image of V)
- $F : P \to P$, $V^{-1} : Q \to P$ which is frobenius semilinear
- $0 \to Q/I_R P \to P/I_R P \to P/Q \to 0$ splits (think of V as the splitting $P/I_R P \to Q/I_R P$)
- P is generated over $W(R)$ by $\operatorname{im} V^{-1}$
- $V^{-1}(v(x)y) = xF(y)$ for $x \in W(R)$

Remark 12.9. $P/I_R P$ is locally free over R, and it follows that $Q/I_R P$ and P/Q are as well.

I'll stop saying "locally free" and just use "free" instead.

Locally, we can get a basis $e_1, \dots, e_d, e_{d+1}, \dots, e_n$ of $P/I_R P$. We have $Q = I_R P + \langle e_{d+1}, \dots, e_n \rangle$. So this is generated by the red things:

$$\begin{array}{ccccccc}
e_1 & e_2 & \cdots & e_d & \color{red}{e_{d+1}} & \color{red}{\cdots} & \color{red}{e_n} \\
\\
\color{red}{pe_1} & \color{red}{pe_2} & & & \color{red}{pe_{d+1}} & & \\
\\
\color{red}{\vdots} & \color{red}{\vdots} & & & \color{red}{\vdots} & &
\end{array}$$

The key observation is that the display is packaged into a single matrix:

$$Fe_j = \sum_i \alpha_{ij} e_i \implies V^{-1}(v(x)e_j) = \sum x\alpha_{ij} e_i \qquad j = 1, \dots, d$$

$$V^{-1} e_j = \sum_i \alpha_{ij} e_i \implies Fe_j = V^{-1}(v(1)e_j) = \sum (p\alpha_{ij}) e_i \qquad j = d+1, \dots, h$$

So all of this data is specified by just the α_{ij}'s. You can write these in block form as $B^{-1} = (\alpha_{ij}) = \begin{bmatrix} u_1 & \underbrace{u_2}_{Q} \end{bmatrix}$, $B = \begin{bmatrix} w_1 \\ w_2 . \end{bmatrix}$. When you have (P, VP, F, V^{-1}), $F \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u & pu_2 \end{bmatrix} \begin{bmatrix} fx \\ fy \end{bmatrix}$ and there's a similar formula for V.

A map $\varphi : (P, Q, F, V^{-1}) \to (P', Q', F', (V')^{-1})$ is a $W(R)$-linear map $P \to P'$ preserving the structure. This implies that φ has the form $\begin{bmatrix} a & vb \\ c & d \end{bmatrix}$. If φ is an isomorphism, $F' = \varphi F \varphi^{-1}$, $(V')^{-1} = \varphi V^{-1} \varphi^{-1}$, and

$$B' = \begin{bmatrix} fa & b \\ pfc & d \end{bmatrix} B \begin{bmatrix} a & vb \\ c & d \end{bmatrix}^{-1}.$$

Definition 12.10. Let $B|_Q$ be the lower right-hand block in B. Define $\overline{B}$ to be the reduction mod p, I_R.

The display is *nilpotent* if $f^n \overline{B} \cdot \dots \cdot f\overline{B} \cdot \overline{B}$ is 0 for some n.

Theorem 12.11. *Let R be a formal $\mathbb{Z}_p$-algebra. Then there is a correspondence*

$$\{\text{formal } p\text{-divisible groups over } R\} \longleftrightarrow \{\text{nilpotent displays over } R\}$$

By Cartier duality, you get:

Corollary 12.12.

$$\left\{\begin{array}{c}\textit{p-divisible groups over } R \textit{ of dimension 1,}\\ \textit{height } h\textit{, formal height} \geq 2\end{array}\right\} \overset{\sim}{\hookleftarrow} \left\{\begin{array}{c}\textit{nilpotent displays over } R\\ \textit{of height } h\textit{, dimension } (h-1)\end{array}\right\}$$

Call the first thing $\mathcal{M}_p(h)_{\geq 2}$ *and the second thing* $\mathrm{Disp}_R^{h,h-1}$.

Let $Wt = \mathbb{Z}[a_0, \dots]$ be the Witt ring. Let A corepresent displays; this is like $Wt^{\otimes h^2}[\det^{-1}]$. Let Γ corepresent isomorphisms $\varphi = \begin{bmatrix} a & vb \\ c & d \end{bmatrix}$ where c has length $h-1$ and d has length 1. You get a Hopf algebroid (A, Γ), and to impose the condition that the displays are nilpotent, you have to complete it (I won't say what we're completing w.r.t.): $(A, \Gamma) \to (\widehat{A}, \widehat{\Gamma}) \simeq \mathcal{M}_p(h)_{\geq 2}$.

Roughly, $\mathrm{Spf}\, R \to \mathcal{M}_{\mathrm{displ}} \simeq \mathcal{M}_p(h)_{\geq 2}$. You want to understand the tangent space. This is a really concrete question.

Let k be a field of characteristic $p > 0$.

$$\{\text{Displays } B + \varepsilon S\} \,/\, \mathrm{Iso}_{k[\varepsilon]/\varepsilon^2} \text{ restricting to 1 over } k \to \left\{\text{Displays } \widetilde{B}/k[\varepsilon]/\varepsilon^2\right\} / \, \mathrm{Iso}_{k[\varepsilon]/\varepsilon^2}$$
$$\to \{\text{Displays } B/k\} \,/\, \mathrm{Iso}_k$$

Say $B = \begin{bmatrix} \alpha & \beta \\ \gamma & v\delta \end{bmatrix} \in \mathrm{Mat}_h(W(\varepsilon k))$.

$$B + \varepsilon S \sim (I + \begin{bmatrix} fa & b \\ pfc & fd \end{bmatrix})(B + \varepsilon S)(I + \begin{bmatrix} a & vb \\ c & d \end{bmatrix})^{-1} = B + \varepsilon s - B \begin{bmatrix} a & vb \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0B \end{bmatrix}$$

This is

$$-B \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} - B \begin{bmatrix} 0 & vb \\ 0 & d \end{bmatrix} + \begin{bmatrix} b\gamma & bv\delta \\ 0 & 0 \end{bmatrix}$$

You can convince yourself that from the first piece, you get all the first $h-1$ columns, from the second piece you get all the last columns $\equiv d \begin{bmatrix} \beta \\ v\delta \end{bmatrix} \pmod{I_R}$ (and the last piece doesn't do anything).

$$\begin{tikzcd} \mathrm{Spec}\, A \arrow[r, "\text{last column mod } I_R"] \arrow[d] & \mathbb{P}^{h-1} \\ \mathrm{Spf}\, \widehat{A} // \mathrm{Spf}\, \widehat{\Gamma} = \mathcal{M}_p(h)_{\geq 2} \arrow[ur, dotted] & \end{tikzcd}$$

For all purposes we can pretend that there's a lift that is an isomorphism on tangent spaces. Suppose we have $\mathrm{Spf}(R) \to \mathrm{Spec}\, A$. When is $\mathrm{Spf}(R) \to \mathcal{M}_p(h)_{\geq 2}$ formally étale? Iff $\mathrm{Spec}\, R$ is formally smooth over $\mathbb{Z}_p$ and it's an isomorphism on tangent spaces. But via our isomorphism on tangent spaces, this is equivalent to $\mathrm{Spf}\, R$ being formally smooth and $\Phi : \mathrm{Spf}\, R \to \mathbb{P}^{h-2}$ being an isomorphism on tangent spaces. This is the same as Φ being formally étale.

To summarize:

Theorem 12.13. *Let R be a formal $\mathbb{Z}_p$-algebra and B is a nilpotent display of height h and dimension $h-1$. Suppose $\Phi : \operatorname{Spf} R \to \operatorname{Spec} A \to \mathbb{P}^{h-1}$ is formally étale. Then there exists an even periodic E_∞-ring E such that*

(1) $E_0 = R$, $E_2 = Q/I_R P$

(2) $\operatorname{Spf} E_0^{\mathbb{C}\mathrm{P}^\infty} = \mathbb{G}^{for}$

Here's an application. Let $h \geq 2$. Consider $R = (\mathbb{Z}[u_1, \ldots, u_{h-1}])^{\widehat{}}_{(p,u_1)}$ and consider the display: $\begin{bmatrix} 0 & \ldots & 0 & 1 \\ & & & [u_{h-1}] \\ & & & \vdots \\ & \mathbb{1} & & [u_2] \\ & & & [u_1] \end{bmatrix}$. Consider $\Phi : \operatorname{Spf} R \to \mathbb{P}^{h-1}$ given by $[1 : u_{h-1} : \cdots : u_1]$; this is the completion of an affine coordinate on $\mathbb{P}^{h-1}$, so it is clearly formally étale. So you get E such that $E_* = R[u^\pm]$.

If you take into account some Galois actions, you get an E_∞-ring $\widetilde{E}$ such that $\widetilde{E} \simeq L_{K(2)\vee\cdots\vee K(n)} E(n)$ and $\widetilde{E}_* = \mathbb{Z}[v_1, \ldots, v_{n-1}, v_n^\pm]^{\widehat{}}_{(p,u_1)}$ where $v_i = u^{p^i - 1} u_i$. This improves on the fact that Morava E-theory is an E_∞-ring.