Operads And Gamma-Homology Of Commutative Rings

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OPERADS AND Γ-HOMOLOGY OF COMMUTATIVE RINGS

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Introduction

In this paper, we construct and investigate the natural homology theory for coherently homotopy commutative dg-algebras, usually known as E_∞-algebras. We call the theory Γ-homology for historical reasons (see, for instance, [3]).

Since discrete commutative rings are E_∞ rings, we obtain by specialization a new homology theory for commutative rings. This special case is far from trivial. It has the following application in stable homotopy theory, which was our original motivation and which will be treated in a sequel to this paper. The obstructions to an E_∞ multiplicative structure on a spectrum lie (under mild hypotheses) in the Γ-cohomology of the corresponding dual Steenrod algebra, just as the obstructions to an A_∞-structure lie in the Hochschild cohomology of that algebra [15].

The Γ-homology of a discrete commutative algebra B can be understood as a refinement of Harrison homology, which was originally defined as the homology of the quotient of the Hochschild complex by the subcomplex generated by nontrivial shuffle products. It is better defined as the homology of a related complex which one obtains by tensoring each term $B^{\otimes n}$ with a certain integral representation V_n of the symmetric group Σ_n, and passing to Σ_n-covariants. Harrison theory works very well in characteristic zero, but not otherwise. A more satisfactory theory necessarily involves the higher homology of the symmetric groups, not only the covariants H_0. Our Γ-homology theory is constructed to do just that. It is furthermore completely different (except in characteristic zero) from André/Quillen homology, which is related to a completely different class of problems. (Polynomial algebras are acyclic for André/Quillen theory by its construction; but they are not generally free E_∞-algebras, and their Γ-homology is generally non-zero.)

There are two further significant generalizations. First, there is a *cyclic* variant of the Γ-homology of any E_∞ dg-algebra. This arises very naturally from our construction in §3. The cyclic theory, like standard Γ-cohomology, is connected with an obstruction-theoretic problem. A full account will appear elsewhere. Second, the domain of definition can be widened from the abelian situation of dg-algebras to the case of spectra in stable homotopy theory, so that one can define the Γ-homology of an E_∞ ring spectrum. This is analogous to extending Hochschild homology to topological Hochschild homology (which includes Mac Lane homology as a special

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case). The generalization to spectra is not difficult, but we have postponed the details to another sequel.

We mention that E_∞ homology was invented independently by the first author and by Waldhausen during the 1980's, and outlined in various lectures, including a plenary lecture by the first author to the Adams Memorial meeting in 1990. Some details subsequently appeared in the second author's thesis [19] and a preprint [16]. Meanwhile Kriz [13] and later Basterra [4] were developing, by methods very different from ours, an E_∞ cohomology theory for ring spectra, which is extremely likely to be equivalent to ours.

The paper is organised as follows. Section 1 contains material on operads. In Section 2 we introduce complexes over operads and define the realization of such a complex (2.8). A cyclic variant of the construction is also given (2.9). Section 3 covers the most important case of realization, namely the Γ-cotangent complex of an E_∞ algebra. This section also contains the definitions of Γ-homology (3.2) and cyclic Γ-homology (3.10), and a transitivity theorem (3.4). (The proof of a key acyclicity lemma is deferred to Appendix A.) In Section 4 we further justify our constructions by showing that their A_∞ analogues lead to Hochschild and cyclic homology of associative algebras. Section 5 is devoted to the special case of Γ-homology of discrete commutative algebras. It is shown that for pairs of $\mathbb{Q}$-algebras Γ-homology coincides with André/Quillen homology (5.6) and an example is given to show the theories are different in general. The final section describes a product in Γ-cohomology of a discrete commutative algebra.

1. Operads, cyclic operads and cofibrancy

We work in the category of chain complexes (dg-modules) over a commutative ground ring K. (We might equally well, of course, have chosen simplicial modules.) Our principal definitions use Getzler and Kapranov's theory [8] of cyclic operads, but we require Markl's non-unital version which is described in [9].

1.1 Operads. Let $\mathbb{S}$ denote the category of finite sets S and isomorphisms of sets, $\mathbb{S}_+$ the subcategory of non-empty sets, and $\mathbb{S}^1$ the category of based finite sets and isomorphisms. (To avoid foundational difficulties, we assume without further mention where necessary that these have been replaced by equivalent small subcategories. Our constructions do not depend upon the choice of subcategory. One can for instance take just one set $\{1, 2, \ldots, n\}$ in $\mathbb{S}$ for each $n \geq 0$, and similarly $\{0, 1, 2, \ldots, n\}$ for each $n \geq 0$ in $\mathbb{S}^1$, so that both categories become disjoint unions of symmetric groups.) An operad $\mathcal{C}$ has objects (chain complexes) $\mathcal{C}_S$ indexed by all finite sets S, isomorphisms $\varphi_* : \mathcal{C}_S \to \mathcal{C}_T$ induced by isomorphisms $\varphi : S \to T$ of sets, and composition maps

$$\circ_t : \mathcal{C}_S \otimes \mathcal{C}_T \to \mathcal{C}_{S \sqcup_t T}$$

for all finite sets S and T, and all elements $t \in T$, where $S \sqcup_t T$ is the deleted sum $S \sqcup (T \setminus \{t\})$; these data must satisfy standard conditions of functoriality and associativity of composition. The induced isomorphisms give a left action of the symmetric group Σ_S of automorphisms of S on $\mathcal{C}_S$. One thinks of $\mathcal{C}_S$ as a parameter space of operations (in the sense of universal algebra) with inputs labelled by S, and a single output; the induced isomorphisms correspond to permutation of inputs, and the composition $\circ_t$ to substitution of the output of $\mathcal{C}_S$ for the input labelled t in

$\mathcal{C}_T$. The operad $\mathcal{C}$ is said to be E_∞ if, for each $S \in \mathbb{S}$, the complex $\mathcal{C}_S$ is contractible and Σ_S-free. It is obviously sufficient to check this for $S = \{1, 2, \dots, n\}$, for all $n \geq 0$.

The standard example of an E_∞ operad is $\mathcal{D}$, in which $\mathcal{D}_S$ is the nerve of the category $\mathbb{S}/S$ of isomorphisms of finite sets over S. Composition in $\mathcal{D}$ is induced by the deleted sum functor in $\mathbb{S}$.

1.2 Cyclic operads. A cyclic operad can be defined as an operad with extra structure (a Σ_{n+1}-action on $\mathcal{C}_{\{1,2,\dots,n\}}$) which makes composition symmetric by putting the 'output' variable 0 on the same footing as the n 'inputs': see [8]. Then it is clearly desirable to change the notation, and denote by $\mathcal{C}_{S\sqcup 0}$ what was previously denoted $\mathcal{C}_S$. Confusion can arise, so we stress that from now on we shall use the 'cyclic' convention, and include the output in the labelling set.

It seems best to define cyclic operads directly. A *cyclic operad* is a functor $\mathcal{E}$ from the category $\mathbb{S}_+$ of non-empty finite sets to the category of chain complexes, together with composition operations

$$\circ_{s,t} : \mathcal{E}_S \otimes \mathcal{E}_T \to \mathcal{E}_{S\sqcup_{s,t}T}$$

for all finite sets S, T with at least two elements, and all choices of $s \in S$, $t \in T$. Here $S \sqcup_{s,t} T$ denotes the deleted sum $(S \setminus \{s\}) \sqcup (T \setminus \{t\})$, and $\circ_{s,t}$ is required to be a natural transformation of functors from $\mathbb{S}^1 \times \mathbb{S}^1$ to chain complexes having the associativity property

$$\circ_{s,t}(1 \otimes \circ_{t',u}) \quad = \quad \circ_{t',u}(\circ_{s,t} \otimes 1)$$

for $s \in S$, $t, t' \in T$, $t \neq t'$, $u \in U$, and the symmetry property

$$\tau_* \cdot \circ_{s,t} \quad = \quad \circ_{t,s} \cdot \tau_\otimes$$

where $\tau_* : \mathcal{E}_{S\sqcup_{s,t}T} \approx \mathcal{E}_{T\sqcup_{t,s}S}$ is induced by the isomorphism of sets and $\tau_\otimes : \mathcal{E}_S \otimes \mathcal{E}_T \approx \mathcal{E}_T \otimes \mathcal{E}_S$ interchanges factors and introduces the usual sign.

Some further notation will be needed. The composition

$$\circ_{s,t} : \mathcal{E}_S \otimes \mathcal{E}_T \to \mathcal{E}_V$$

is associated with a *partition* of V into two subsets $S \setminus \{s\}$ and $T \setminus \{t\}$. Conversely, let $V = P \sqcup Q$ be any partition of V into two non-empty sets. We can define the associated composition by writing P^1 and Q^2 for the disjoint unions $P \sqcup \{1\}$ and $Q \sqcup \{2\}$, and taking

$$\circ_{12} : \mathcal{E}_{P^1} \otimes \mathcal{E}_{Q^2} \to \mathcal{E}_V \ .$$

1.3 E_∞ cyclic operads. We call $\mathcal{E}$ an E_∞ *cyclic operad* if for all $S \in \mathbb{S}_+$ the complex $\mathcal{E}_S$ is contractible and Σ_S-free. It suffices to check this for $S = \{0, 1, \dots, n\}$ for all $n \geq 1$. The operad $\mathcal{D}$ defined as in 1.1 is an E_∞ cyclic operad, the composition again being induced by the deleted sum functor.

1.4 Cofibrant operads. We adopt the notation introduced in 1.2 for adding new points to a set: S^1, S^2 and S^{12} are to denote $S \sqcup \{1\}$, $S \sqcup \{2\}$ and $S \sqcup \{1, 2\}$ respectively. For each partition $V = S \sqcup T$ of V we have a composition map $\circ_{21} : \mathcal{E}_{S^2} \otimes \mathcal{E}_{T^1} \to \mathcal{E}_V$. In a cofibrant operad, provided S and T each have more than one element, we want this map to be the inclusion of a *face* of $\mathcal{E}_V$, so we require it to be an (equivariant) cofibration; and we require faces to intersect only in faces of faces. This leads to the following.

Definition. Let $\partial\mathcal{E}_V$ denote the coequalizer of the maps

$$\bigoplus_{V=P\sqcup Q\sqcup R} \mathcal{E}_{P^2}\otimes\mathcal{E}_{Q^{12}}\otimes\mathcal{E}_{R^1} \quad \underset{1\otimes\circ_{21}}{\overset{\circ_{21}\otimes 1}{\rightrightarrows}} \quad \bigoplus_{V=S\sqcup T}\mathcal{E}_{S^2}\otimes\mathcal{E}_{T^1}$$

where the sums are indexed by partitions of V into subsets of which S and T, and hence P and R, have at least two elements each. Associativity of composition implies that $\circ_{21}$ induces a Σ_V-equivariant map $\partial\mathcal{E}_V\to\mathcal{E}_V$, which we call the *inclusion of the boundary* . The cyclic operad $\mathcal{E}$ is *cofibrant* if

(1) for every V the inclusion of the boundary is a Σ_V-equivariant cofibration;
(2) there is a given augmentation $\varepsilon:\mathcal{E}_E\to K$ when E has exactly two elements, invariant with respect to induced maps φ_*, such that for every partition of a set $V=W\sqcup\{w\}$ into a set and a singleton, the mapping

$$\mathcal{E}_{W^1}\otimes\mathcal{E}_{\{2,w\}}\xrightarrow{1\otimes\varepsilon}\mathcal{E}_{W^1}\otimes K=\approx\mathcal{E}_{W^1}\xrightarrow{\psi_*}\mathcal{E}_V\ ,$$

where ψ is the evident isomorphism $W^1\to W\sqcup\{w\}=V$, coincides with mapping given by the composition $\circ_{1,2}$.

Cofibrant non-cyclic operads are defined in a completely analogous way.

1.5 The E_∞ tree operad $\mathcal{T}$. We now construct a cofibrant E_∞ cyclic operad. The cyclic operad $\mathcal{D}$ will not do: the faces of $\mathcal{D}_S$ intersect in unacceptably large subcomplexes, so that $\partial\mathcal{D}_S\to\mathcal{D}_S$ is not injective. On the other hand, we can form another cyclic operad by taking $\mathcal{E}_S$ to be the space of trees [17] with ends labelled by the set S, and $\circ_{s,t}$ to be the operation of grafting the end labelled s to the end labelled t to produce a new edge of length 1 . This operad has every $\mathcal{E}_S$ contractible, and it is cofibrant; but it is not an E_∞ operad because Σ_n does not act freely on $\mathcal{E}_n$. (In the realm of A_∞ operads, which are indexed by ordered finite sets and have no Σ_n action, there is a corresponding operad in which the objects are the complexes of cyclically-labelled trees in the plane: it is the analogue of the topological operad of Stasheff polyhedra – see [5].)

By combining the two constructions we obtain a cofibrant E_∞ cyclic operad, the *tree operad* $\mathcal{T}$, as follows. We take $\mathcal{T}_S$ to be the chain (bi)complex associated with the bisimplicial set in which a (k,l)-bisimplex consists of a k-simplex of the nerve of the category $\mathbb{S}/S$

$$\begin{array}{ccccccc} S_k & \xrightarrow{\varphi_k} & S_{k-1} & \xrightarrow{\varphi_{k-1}} & \dots & \xrightarrow{\varphi_1} & S_0\\ &&&&&&\downarrow\\ &&&&&& S\end{array}$$

together with an l-simplex of the space $\tilde{T}_{S_k}$ of trees labelled by the set S_k; and the simplicial operators are defined in the obvious way. The composition maps $\circ_{s,t}$ in $\mathcal{T}$ are defined by using the deleted sum functor in the category $\mathbb{S}$ and the grafting of trees, as above. The operad $\mathcal{T}$ inherits the E_∞ property of $\mathcal{D}$. We show that it also has the cofibrancy of the operad of trees. To show that the inclusion of the boundary $\partial\mathcal{T}_V$ is an equivariant cofibration, we have to verify that it is induced by an injective map of bisimplicial sets, and that the group of automorphisms of V acts freely on its complement. The freeness follows from the freeness on $\mathcal{D}_V$.

For injectivity, the essence is that a simplex lies in the face corresponding to a decomposition $V = S \sqcup T$ if and only if it consists of trees in which there is an internal edge, of maximal length, which separates the labels S from the labels T; and therefore two faces meet only where two specified edges have maximal length, which is a face of a face (or is empty, as appropriate).

2. Algebras, modules and realization

2.1 Algebras and modules over an operad. Let $\mathcal{C}$ be a cyclic operad, and K the ground ring, which is commutative with unit element.

Definition. An *algebra* over $\mathcal{C}$ is a chain complex (of K-modules) A together with structural maps $\mu_S : \mathcal{C}_{S^0} \otimes A^{\otimes S} \to A$ for all (non-empty) sets $S \in \mathbb{S}_+$ which are natural in S and satisfy the usual condition

$$\mu_{S^1}(1 \otimes \mu_T) = \mu_{S \sqcup T}(\circ_{10} \otimes 1)$$

of equality of maps $\mathcal{C}_{S^{01}} \otimes \mathcal{C}_{T^0} \otimes A^{\otimes(S \sqcup T)} \to A$.

By way of explanation we note that 0 has been adjoined to S and T as the 'output variable' for the operad. The element 1 in S^{01} is a dummy label associated with the partition $S \sqcup T$, as introduced at the end of 1.2.

When the smallest model is chosen for $\mathbb{S}_+$, which is the disjoint union of the symmetric groups Σ_n for $n \geq 1$, the naturality condition in the definition simply means that μ_n is equivariant and so defines a map $\mathcal{C}_{n+1} \otimes_{\Sigma_n} A^{\otimes n} \to A$, where Σ_n acts on $\mathcal{C}_{n+1}$ on the left (fixing the output label) and on $A^{\otimes n}$ on the right.

Definition. An *A-module* over $\mathcal{C}$, when A is a $\mathcal{C}$-algebra as above, is a chain complex M together with structural maps $\nu_S : \mathcal{C}_{S^{01}} \otimes A^{\otimes S} \otimes M \to M$ which are natural in S and satisfy the usual module conditions

$$\nu_S(1 \otimes \nu_T) = \nu_{S \sqcup T}(\circ_{10} \otimes 1)$$

as maps $\mathcal{C}_{S^{01}} \otimes \mathcal{C}_{T^{01}} \otimes A^{\otimes(S \sqcup T)} \otimes M \to M$, and

$$\nu_{S^2}(1 \otimes \mu_T) = \nu_{S \sqcup T}(\circ_{20} \otimes 1)$$

as maps $\mathcal{C}_{S^{012}} \otimes \mathcal{C}_{T^0} \otimes A^{\otimes(S \sqcup T)} \otimes M \to M$.

The above algebras and modules are non-unital. This defect will be remedied in the next section.

2.2 Algebras and modules over an E_∞ operad.
From now on it is a standing assumption (except where the reverse is stated) that all operads are cyclic and E_∞. An algebra A over such an operad $\mathcal{C}$ will be called an E_∞ algebra. Since $\mathcal{C}$ is automatically augmented over the standard commutative algebra operad, the ground ring K is an E_∞ algebra.

For the purposes of this paper, it suffices to define subalgebras and submodules in a naive way as chain subcomplexes which are closed under the appropriate operad action. If A is a subalgebra of B over $\mathcal{C}$, there is an inclusion homomorphism $A \to B$, and we call B an A-algebra over $\mathcal{C}$. We shall usually work with K-algebras, where K is the ground ring regarded as an algebra over $\mathcal{C}$. The unit element of K then serves as a unit for the algebra. When considering modules over a K-algebra A, we require the induced K-module structure to be the standard, strict one.

2.3 Cyclic and non-cyclic complexes over an operad.

We now aim to construct the *homotopical cotangent complex* $\mathcal{K}(B/A;M)$ when B is an E_∞ K-algebra, A a K-subalgebra, and M a B-module. (Although very different in appearance and in construction, this will play in our theory the rôle analogous to that played in André-Quillen theory by the cotangent complex of [1], [14]).

Our cotangent complex will be a filtered object obtained by glueing together the objects $\mathcal{C}_{V^0} \otimes B^{\otimes V} \otimes M$, where V runs through the category $\mathbb{S}_+$. (There is also a cyclic version.) Conceptually it resembles the realization of a simplicial object, or the analogue described in [18]. Because the realization sometimes has to be applied to species other than the standard $B^{\otimes V} \otimes M$, it is worthwhile to formulate a definition of the kind of general object which can be realized.

2.4 Definition. Let $\mathcal{C}$ be a cyclic operad. A *cyclic $\mathcal{C}$-complex* is a cofunctor $\mathcal{M}$ from the category $\mathbb{S}_+$ to the category of chain complexes, together with the following further data which specify an action of $\mathcal{C}$ on $\mathcal{M}$: for each composition

$$\circ_{s,t} : \mathcal{C}_S \otimes \mathcal{C}_T \to \mathcal{C}_{S \sqcup_{s,t} T}$$

in $\mathcal{C}$ there is given a formal adjoint

$$\circ^*_{s,t} : \mathcal{C}_S \otimes \mathcal{M}_{S \sqcup_{s,t} T} \to \mathcal{M}_T$$

(to be thought of as a cap product corresponding to the above cup product) which satisfies

(1) the naturality condition
$$\psi^* \circ^*_{s',t'} (\varphi_* \otimes 1) \;=\; \circ^*_{s,t}(1 \otimes (\varphi \sqcup_{s,t} \psi)^*)$$
for all isomorphisms $\varphi : (S,s) \approx (S',s')$, $\psi : (T,t) \approx (T',t')$ in $\mathbb{S}$;

(2) the associativity condition
$$\circ^*_{34}(1 \otimes \circ^*_{12}) \;=\; \circ^*_{34}(\circ_{21} \otimes 1) : \mathcal{C}_{S^{32}} \otimes \mathcal{C}_{R^1} \otimes \mathcal{M}_{R \sqcup S \sqcup T} \to \mathcal{M}_{T^4}$$
for all finite sets R^1, S^{23}, T^4;

(3) the associativity condition
$$\circ^*_{43}(1 \otimes \circ^*_{12}) \;=\; \circ^*_{12}(1 \otimes \circ^*_{43})(\tau \otimes 1) : \mathcal{C}_{T^4} \otimes \mathcal{C}_{R^1} \otimes \mathcal{M}_{R \sqcup S \sqcup T} \to \mathcal{M}_{S^{23}}$$
for all finite sets R^1, S^{23}, T^4, where τ interchanges factors.

There are two associativity conditions above for the same reason as in the definition of non-cyclic operad: there are two types of iterated substitution to be considered.

2.5 Example. Suppose A is an algebra over the operad $\mathcal{C}$, with structural maps $\mu_V : \mathcal{C}_{V^0} \otimes A^{\otimes V} \to A$. Then we can take $\mathcal{M}_S = A^{\otimes S}$, and define $\circ^*_{s,t}$ to be

$$\mathcal{C}_S \otimes A^{\otimes (S \sqcup_{s,t} T)} \;\approx\; \mathcal{C}_S \otimes A^{\otimes (S \setminus \{s\})} \otimes A^{\otimes (T \setminus \{t\})} \xrightarrow{\mu_{S\setminus\{s\}} \otimes 1} A \otimes A^{\otimes (T \setminus \{t\})} \approx A^{\otimes T}$$

giving a cyclic $\mathcal{C}$-complex.

Now we need the non-cyclic version, which is slightly more complicated because the indexing sets have basepoints and there are, as in the definition of operads, correspondingly more cases to consider. (The basepoint may be in any subset of a partition.) We consistently write 0 for the basepoint, so that a typical based finite set is S^0, where S is an object of $\mathbb{S}$.

2.6 Definition. Let $\mathcal{C}$ be a cyclic operad. A (non-cyclic) $\mathcal{C}$-*complex* assigns to each based finite set S^0 a chain complex $\mathcal{M}_{S^0}$ depending cofunctorially upon S^0, and to each composition

$$\circ_{0,1} : \mathcal{C}_{S^0} \otimes \mathcal{C}_{T^{01}} \to \mathcal{C}_{(S \sqcup T)^0}$$

in $\mathcal{C}$ a pair of formal adjoints

$$\circ^*_{0,1} : \mathcal{C}_{S^0} \otimes \mathcal{M}_{(S \sqcup T)^0} \to \mathcal{M}_{T^{01}}$$

and

$$\circ^*_{1,0} : \mathcal{C}_{T^{01}} \otimes \mathcal{M}_{(S \sqcup T)^0} \to \mathcal{M}_{S^0}$$

satisfying the analogues of the naturality and associativity conditions (1)–(3) of 2.3. The asymmetry in this definition arises because the composition $\circ_{0,1}$ corresponds to a partition of $V^0 = (S \sqcup T)^0 = S \sqcup T^0$ into a subset which does not contain the basepoint and one which does. The first of the two adjoints above evaluates over S, and takes values in $\mathcal{M}_{T^{01}}$, where 1 is a dummy label from the partition; the second evaluates over T^0, and takes values in $\mathcal{M}_{S^0}$, where 0 is a new dummy basepoint for S.

2.7 Example. Just as the primary example of a cyclic $\mathcal{C}$-complex arises from an algebra (2.5), so the primary example of a non-cyclic $\mathcal{C}$-complex arises from a module. In definition 2.6 above, let $\mathcal{M}_{S^0} = A^{\otimes S} \otimes M$ where A is a K-algebra over $\mathcal{C}$ and M an A-module; let $\circ^*_{0,1}$ be $\mu_S \otimes 1$, where μ is the algebra structure, and $\circ^*_{1,0}$ be $1 \otimes \nu_T$, where ν is the module structure.

2.8 The realization of a $\mathcal{C}$-complex.

Let $\mathcal{C}$ be a cofibrant cyclic operad, and $\mathcal{M}$ a $\mathcal{C}$-complex. We construct the *realization* $|\mathcal{M}|$ by a process resembling that for realizing a simplicial set. We treat the non-cyclic case in detail, because it is more important for us, then describe the differences in the cyclic case, which is important in cyclic Γ-homology. There are two steps in the construction.

First we construct a complex $|\mathcal{M}|'$. We take a direct sum over all $V^0 = V \sqcup \{0\}$ in $\mathbb{S}^1$ (our category of based sets) having three or more elements

$$\bigoplus_{|V^0| \geq 3} \mathcal{C}_{V^0} \otimes \mathcal{M}_{V^0} \quad .$$

$|\mathcal{M}|'$ is the quotient of this by the following identifications:

(1) for each isomorphism $\varphi : S^0 \approx T^0$ in $\mathbb{S}^1$, and all $x \in \mathcal{C}_{S^0}$, all $m \in \mathcal{M}_{T^0}$

$$\varphi_* x \otimes m \quad \sim \quad x \otimes \varphi^* m \ ;$$

(2) for each partition $V^0 = S \sqcup T^0$ of a set into two subsets (the second containing the basepoint) having at least two elements each, we consider the associated composition (as in 1.2)

$$\circ_{0,1} : \mathcal{C}_{S^0} \otimes \mathcal{C}_{T^{01}} \to \mathcal{C}_{(S \sqcup T)^0} = \mathcal{C}_{V^0}$$

and we define

$$\partial^{S,T} : \quad \mathcal{C}_{S^0} \otimes \mathcal{C}_{T^{01}} \otimes \mathcal{M}_{(S \sqcup T)^0} \quad \longrightarrow \quad \mathcal{C}_{S^0} \otimes \mathcal{M}_{S^0} \quad \oplus \quad \mathcal{C}_{T^{01}} \otimes \mathcal{M}_{T^{01}}$$

by setting

$$\partial^{S,T} \quad = \quad (1 \otimes \circ_{10}^*) \quad \oplus \quad (1 \otimes \circ_{01}^*)(\tau \otimes 1)$$

where τ interchanges the factors $\mathcal{C}_{S^0}$ and $\mathcal{C}_{T^{01}}$ and introduces the usual sign. Then on the component of the boundary $\partial\mathcal{C}_{V^0} \otimes \mathcal{M}_{V^0}$ corresponding to $\circ_{0,1}$ we make identifications by requiring

$$\circ_{01}(x \otimes y) \otimes m \quad \sim \quad \partial^{S,T}(x \otimes y \otimes m)$$

for all $x \in \mathcal{C}_{S^0}$, $y \in \mathcal{C}_{T^{01}}$, $m \in \mathcal{M}_{V^0}$.

The complex $|\mathcal{M}|'$ thus defined is a quotient of

$$\bigoplus_{n \geq 2} \mathcal{C}_{n+1} \otimes_{\Sigma_n} \mathcal{M}_{n+1}$$

where $n+1$ denotes the $(n+1)$-element based set $\{0, 1, \ldots, n\}$. This is because the identifications (1) imply that it suffices to take one indexing set of each size, and pass to the quotient by the action of Σ_n. We can define the *skeletal filtration* of $|\mathcal{M}|'$ by defining the *k-skeleton* to be the image of $\bigoplus_{2 \leq n \leq k} \mathcal{C}_{n+1} \otimes \mathcal{M}_{n+1}$. Just as in the standard simplicial realization construction, the identifications (2) satisfy compatibility conditions which guarantee that the kth filtration quotient of $|\mathcal{M}|'$ is isomorphic to $(\mathcal{C}_{k+1}/\partial\mathcal{C}_{k+1}) \otimes_{\Sigma_k} \mathcal{M}_{k+1}$.

We now describe the second step in the construction, which incorporates the bottom filtration stage $\mathcal{M}_2$. (Since $\mathcal{C}_2$ is contractible for an E_∞ operad, and Σ_1 is trivial, $\mathcal{M}_2$ is quasi-isomorphic to the expected bottom filtration stage $\mathcal{C}_{1+1} \otimes_{\Sigma_1} \mathcal{M}_{1+1}$.)

Let V^0 be any based set in $\mathbb{S}^1$ having three or more elements. Take any $v \in V$, and write T_v for $V \setminus \{v\}$. The partition $V^0 = \{v\} \sqcup T_v$ has an associated composition

$$\circ_{0,1} : \mathcal{C}_{\{v,0\}} \otimes \mathcal{C}_{T_v^{01}} \to \mathcal{C}_{V^0}$$

and action

$$\circ_{1,0}^* : \mathcal{C}_{T_v^{01}} \otimes \mathcal{M}_{V^0} \to \mathcal{M}_{\{v,0\}} \ .$$

The standard isomorphisms $T_v^{01} = V \setminus \{v\} \sqcup \{0, 1\} \approx V^0$ and $\{v, 0\} \approx \{0, 1\}$, taking v to 1 in each case, convert this action into a map

$$\varepsilon_v : \mathcal{C}_{V^0} \otimes \mathcal{M}_{V^0} \to \mathcal{M}_{\{0,1\}} = \mathcal{M}_2 \ .$$

If instead of $v \in V$ we select $0 \in V^0$, the other action map $\circ_{0,1}^*$ (defined in 2.6) yields in identical fashion a map

$$\varepsilon_0 : \mathcal{C}_{V^0} \otimes \mathcal{M}_{V^0} \to \mathcal{M}_{\{0,1\}} = \mathcal{M}_2 \ .$$

Let $\varepsilon = \varepsilon_0 - \sum_{v \in V} \varepsilon_v$. This defines, for each V^0 in $\mathbb{S}^1$ having three or more elements, a map $\mathcal{C}_{V^0} \otimes \mathcal{M}_{V^0} \to \mathcal{M}_2$. One can check immediately that the naturality and associativity conditions in the definition of a $\mathcal{C}$-complex imply that these maps are compatible with the identifications used in the construction of $|\mathcal{M}|'$. Therefore we have a well-defined map $\varepsilon : |\mathcal{M}|' \to \mathcal{M}_2$. The final realization $|\mathcal{M}|$ is defined to be the cofibre of ε. This completes the construction of the realization in the non-cyclic case.

2.9 Realization in the cyclic case.

Now let $\mathcal{M}$ be a cyclic $\mathcal{C}$-complex, where $\mathcal{C}$ is a cofibrant cyclic operad. We construct the *cyclic realization* $|\mathcal{M}|_{\mathrm{cy}}$ by modifying the construction of 2.8 as follows. We begin with a sum indexed by all V in our category $\mathbb{S}$ of unbased finite sets containing at least three elements

$$\bigoplus_{|V|\geq 3} \mathcal{C}_V \otimes \mathcal{M}_V \ .$$

We alter the identifications to take account of the extra symmetry available in that there is now no basepoint: they now read

(1) for each isomorphism $\varphi : S \approx T$ in $\mathbb{S}$, and all $x \in \mathcal{C}_S$, all $m \in \mathcal{M}_T$

$$\varphi_* x \otimes m \quad \sim \quad x \otimes \varphi^* m \ .$$

(2) for each partition $V = S \sqcup T$ of a set into two subsets having at least two elements each, and we define

$$\partial^{S,T} : \quad \mathcal{C}_{S^2} \otimes \mathcal{C}_{T^1} \otimes \mathcal{M}_{(S\sqcup T)} \quad \longrightarrow \quad \mathcal{C}_{S^2} \otimes \mathcal{M}_{S^2} \quad \oplus \quad \mathcal{C}_{T^1} \otimes \mathcal{M}_{T^1}$$

by setting

$$\partial^{S,T} \quad = \quad (1 \otimes \circ_{12}^*) \quad \oplus \quad (1 \otimes \circ_{21}^*)(\tau \otimes 1)$$

where τ interchanges the factors $\mathcal{C}_{S^2}$ and $\mathcal{C}_{T^1}$ and introduces the usual sign. Then on the component of the boundary $\partial \mathcal{C}_V \otimes \mathcal{M}_V$ corresponding to $\circ_{2,1}$ we make identifications by requiring

$$\circ_{21}(x \otimes y) \otimes m \quad \sim \quad \partial^{S,T}(x \otimes y \otimes m)$$

for all $x \in \mathcal{C}_{S^2}$, $y \in \mathcal{C}_{T^1}$, $m \in \mathcal{M}_V$.

We have now completed the description of the first stage, which we denote $|\mathcal{M}|'_{\mathrm{cy}}$, of the cyclic realization.

The identifications above mean in effect that $|\mathcal{M}|'$ is a quotient of

$$\bigoplus_{n\geq 2} \mathcal{C}_{n+1} \otimes_{\Sigma_{n+1}} \mathcal{M}_{n+1}$$

where Σ_{n+1} is the group of permutations of $\{0, 1, \dots , n\}$.

In analogy with 2.8 we now expect to define a map

$$\varepsilon : \ |\mathcal{M}|'_{\mathrm{cy}} \longrightarrow \mathcal{C}_2 \otimes_{\Sigma_2} \mathcal{M}_2$$

the cofibre of which would be $|\mathcal{M}|_{\mathrm{cy}}$. In actual fact a sign intervenes in the representation, and we have to replace $\mathcal{C}_2$ by a different contractible free Σ_2-complex.

The nerve of the category of isomorphisms of two-element sets is a model for the classifying space $B\Sigma_2$, and the nerve of the category of isomorphisms of ordered two-element sets is its universal cover $E\Sigma_2$. Let V be any set in $\mathbb{S}$ having three or more elements. Take any $v \in V$, and write T_v for $V \setminus \{v\}$. As in 2.8 we have a composition

$$\circ_{0,1} : \mathcal{C}_{\{v,0\}} \otimes \mathcal{C}_{T_v^1} \to \mathcal{C}_V$$

and action

$$\circ_{1,0}^* : \mathcal{C}_{T_v^1} \otimes \mathcal{M}_V \to \mathcal{M}_{\{v,0\}} \ .$$

Using the isomorphism $T_v^1 \approx V$, we obtain from the adjoint $\circ_{1,0}^*$ a map which we shall denote

$$\circ_{v,0}^* : \mathcal{C}_V \otimes \mathcal{M}_V \to \mathcal{M}_{\{v,0\}} \ .$$

As $\{0, v\}$ is an ordered two-element set, we can regard it as a chain of $E\Sigma_2$. We define

$$\tilde{\varepsilon} : \mathcal{C}_V \otimes \mathcal{M}_V \to E\Sigma_2 \otimes \mathcal{M}_{\{v,0\}}$$

by setting

$$\tilde{\varepsilon}(x) = \sum_{v \in V} \left(\{0, v\} \otimes \circ_{v,0}^*(x)\right) \ .$$

This does not yet respect the identifications defining $|\mathcal{M}|'_{\mathrm{cy}}$. But if we denote by $\tilde{\mathcal{M}}_2$ the complex $\mathcal{M}_2$ with its Σ_2-structure twisted by the sign representation, then $\tilde{\varepsilon}$ composes with the quotient map to give a well-defined map

$$\varepsilon : \ |\mathcal{M}|'_{\mathrm{cy}} \to E\Sigma_2 \otimes_{\Sigma_2} \tilde{\mathcal{M}}_2 \ .$$

We finally define the cyclic realization $|\mathcal{M}|_{\mathrm{cy}}$ to be the cofibre of this map.

2.10 Remarks. (1) The sign in the last stage of the above construction is needed to ensure cancellation of the unwanted contributions from the two dummy labels in a partition as in identification 2.9(2) above.

(2) There is a natural map $|\mathcal{M}| \to |\mathcal{M}|_{\mathrm{cy}}$ induced by the levelwise quotient maps $\mathcal{C}_{n+1} \otimes_{\Sigma_n} \mathcal{M}_{n+1} \to \mathcal{C}_{n+1} \otimes_{\Sigma_{n+1}} \mathcal{M}_{n+1}$, which are well-behaved with respect to the identifications in the construction.

2.11 Uniqueness of E_∞ realization.

We now prove that the homotopy type of the realization $|\mathcal{M}|$ or $|\mathcal{M}|_{\mathrm{cy}}$ does not depend upon the cofibrant cyclic operad $\mathcal{C}$ used to construct it, provided that $\mathcal{C}$ is E_∞. The proof uses the standard idea of comparison of resolutions.

Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be E_∞ cyclic operads, with $\mathcal{C}$ cofibrant. Then there is a map $\mathcal{C} \to \mathcal{D}$ of cyclic operads, and it is unique up to homotopy.*

Proof. We construct Σ_{n+1}-equivariant maps $\varphi_{n+1} : \mathcal{C}_{n+1} \to \mathcal{D}_{n+1}$, commuting with all composition maps, by using induction on n. We note first of all that the unit axiom (1.4(2)) for E_∞ operads means that φ_{n+1} will always commute with compositions with $\mathcal{C}_2$ and $\mathcal{D}_2$, since the axiom reduces this to the naturality property. Suppose by inductive hypothesis that we have equivariant φ_{k+1} for all $k < n$, commuting with compositions as far as this makes sense. We have to define φ_{n+1}. The boundary $\partial\mathcal{C}_{n+1}$ is by 1.4 a sum of copies of $\mathcal{C}_{i+1} \otimes \mathcal{C}_{j+1}$ with $2 \le i, j$ and $i + j = n + 1$, amalgamated along $\mathcal{C}_{i+1} \otimes \partial\mathcal{C}_{j+1} \ \cup \ \partial\mathcal{C}_{i+1} \otimes \mathcal{C}_{j+1}$. The maps $\varphi_{i+1} \otimes \varphi_{j+1}$ therefore induce a map $\partial\mathcal{C}_{n+1} \to \mathcal{D}_{n+1}$, equivariant with respect to the induced action of Σ_{n+1}. Since $\mathcal{C}$ is cofibrant and $\mathcal{D}_{n+1}$ is contractible, this map extends to a Σ_{n+1}-equivariant map $\mathcal{C}_{n+1} \to \mathcal{D}_{n+1}$, which by construction retains the compatibility with compositions. Since the induction starts automatically with $n = 1$, where the boundary is empty, the inductive proof of existence is complete. The proof of homotopy uniqueness is similar. $\square$

2.12 Proposition. *If $\mathcal{M}$ is a complex over one E_∞ cyclic operad, then it is a complex over every cofibrant cyclic E_∞ operad, and the homotopy type of the realization $|\mathcal{M}|$ (or $|\mathcal{M}|_{\mathrm{cy}}$, in the cyclic case) is independent of the cyclic cofibrant E_∞ operad used to construct it.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be cyclic E_∞ operads, with $\mathcal{C}$ cofibrant. By 2.11 there is a map of operads, unique up to homotopy, from $\mathcal{C}$ to $\mathcal{D}$. If $\mathcal{M}$ is a $\mathcal{D}$-complex, such a map induces the structure (unique up to homotopy) of a $\mathcal{C}$-complex on $\mathcal{M}$.

Suppose now that $\varphi : \mathcal{C} \to \mathcal{D}$ is a map of E_∞ operads. We show by induction on k that φ_{k+1} is a homotopy equivalence of pairs $(\mathcal{C}_{k+1}, \partial\mathcal{C}_{k+1}) \to (\mathcal{D}_{k+1}, \partial\mathcal{D}_{k+1})$. This is certainly true for $k = 1, 2$, where the spaces are contractible and the boundaries are empty. Suppose it is true for $k < n$. The assembly of $\partial\mathcal{C}_{n+1}$ and $\partial\mathcal{D}_{n+1}$ from cofibrations of lower spaces in the operads (as in the proof of 2.11) implies that φ_{n+1} restricts to a homotopy equivalence $\partial\mathcal{C}_{n+1} \to \partial\mathcal{D}_{n+1}$. But then $\mathcal{C}_{n+1}$ and $\mathcal{D}_{n+1}$ are contractible, and the inclusions of the boundaries are cofibrations, so φ_{n+1} is a homotopy equivalence of pairs.

When $\mathcal{M}$ has the $\mathcal{C}$-structure induced by the map φ, there is a skeleton-preserving induced map $|\varphi| : |\mathcal{M}|_{\mathcal{C}} \to |\mathcal{M}|_{\mathcal{D}}$ between the realizations constructed using the two different operads. On quotients of adjacent skeleta, $|\varphi|$ induces a map

$$(\mathcal{C}_n/\partial\mathcal{C}_n) \otimes_{\Sigma_n} \mathcal{M}_n \xrightarrow{\varphi_n \otimes 1} (\mathcal{D}_n/\partial\mathcal{D}_n) \otimes_{\Sigma_n} \mathcal{M}_n$$

which is a homotopy equivalence because φ_n has been shown to be a homotopy equivalence of free Σ_n-complexes. By induction and direct limit, $|\varphi|$ is a homotopy equivalence. Hence $|\mathcal{M}|$ is independent of the cofibrant E_∞ operad used. A similar proof works in the cyclic case. $\square$

2.13 The homology of the realization.

Proposition.

(1) *Let $\mathcal{M}$ be a $\mathcal{C}$-complex, where $\mathcal{C}$ is an E_∞ operad. Then there is a homology spectral sequence*

$$E^1_{p-1,q} \approx H_q(E\Sigma_p \otimes_{\Sigma_p} (V_p \otimes \mathcal{M}_{p+1})) \Longrightarrow H_{p+q-1}(|\mathcal{M}|)$$

where V_p is the representation of Σ_p on the homology of the tree space T_p, and Σ_p acts diagonally on $V_p \otimes \mathcal{M}_{p+1}$.

(2) *When $\mathcal{M}$ is a cyclic $\mathcal{C}$-complex there is a corresponding homology spectral sequence in the form*

$$E^1_{p-1,q} \approx H_q(E\Sigma_{p+1} \otimes_{\Sigma_{p+1}} (V'_p \otimes \mathcal{M}_{p+1})) \Longrightarrow H_{p+q-1}(|\mathcal{M}|_{\mathrm{cy}})$$

where V'_p is the integral representation of Σ_{p+1} on the homology of T_p.

Proof. The spectral sequence obtained from the skeletal filtration of $|\mathcal{M}|$ or $|\mathcal{M}_{\mathrm{cy}}|$ is independent of the particular cofibrant E_∞ operad used in the construction. Choosing the E_∞ tree operad of 1.5 leads to the E^1 terms given above.

The $E^1_{p-1,*}$ term is in effect the hyperhomology of the group Σ_p or Σ_{p+1} with coefficients in the complex $V_p \otimes \mathcal{M}_{p+1}$ or $V'_p \otimes \mathcal{M}_{p+1}$, and the differential $d^1_{p,*}$ can be expressed in terms of transfer in group hyperhomology and the boundary in $\mathcal{M}$.

Naturally there are the hyperhomology spectral sequence

$$H_s(\Sigma_p; V_p \otimes H_t(\mathcal{M}_{p+1})) \Longrightarrow E^1_{p-1,s+t}$$

and its analogue for the second case, available as subsidiary spectral sequences for calculating the E^1-terms.

It is known that the representation V_p is isomorphic over $\mathbb{Z}$ to $\mathrm{Hom}(\mathrm{Lie}_p, \mathbb{Z}[-1])$, where Lie_p is the Lie representation and $\mathbb{Z}[-1]$ the sign representation. The module V'_p is related to V_p and therefore to the Lie representations by a short exact sequence

$$0 \to V_{p+1} \to \mathrm{Ind}_{\Sigma_p}^{\Sigma_{p+1}} V_p \to V'_p \to 0\ .$$

The complex character of V'_p is calculated in [17]. $\square$

3. The Γ-cotangent complex and the transitivity theorem

3.1 Introduction. We now apply the general theory of §2 to the case we are really interested in: the construction of the Γ-cotangent complex $\mathcal{K}(B/A\,; M)$ when A is a subalgebra of the E_∞ differential graded algebra B, and M is any B-module. In the construction we shall use any cofibrant E_∞ cyclic operad $\mathcal{C}$, such as the tree operad $\mathcal{T}$ of 1.5: the result is independent, up to quasi-isomorphism, of the choice. It is no real loss of generality to assume that A, B and M are flat or even projective over the ground ring K: the structure of algebra or module over a cofibrant operad is homotopy invariant, so that one can replace these objects by projective resolutions; and our realizations are homotopy invariant constructions, so the choice of projective resolution makes no difference to the result. Similarly we may assume that $A \subset B$ is a cofibration.

We also prove a flat base-change result, showing that $\mathcal{K}(B/A\,; M)$ is essentially independent of the ground ring.

The notation of the above introduction will be used throughout the section.

Let $\mathcal{K}$ be the following non-cyclic $\mathcal{C}$-complex, which was mentioned in 2.7. For each based finite set $V^0 = V \sqcup \{0\}$ in $\mathbb{S}^1$ we put

$$\mathcal{K}_{V^0} = A^{\otimes V} \otimes M\ .$$

For every partition $V^0 = S \sqcup T^0$ of V^0 into nonempty sets we define

$$\circ^*_{0,1} : \mathcal{C}_{S^0} \otimes \mathcal{K}_{(S \sqcup T)^0} \to \mathcal{K}_{T^{10}}$$

by using the algebra structure map μ

$$\circ^*_{0,1} = \mu_S \otimes 1 \otimes 1 : \quad \mathcal{C}_{S^0} \otimes A^{\otimes S} \otimes A^{\otimes T} \otimes M \longrightarrow A \otimes A^{\otimes T} \otimes M\ ,$$

and we define

$$\circ^*_{1,0} : \mathcal{C}_{T^{10}} \otimes \mathcal{K}_{(S \sqcup T)^0} \to \mathcal{K}_{S^0}$$

by using the module structure map ν

$$\circ^*_{1,0} = 1 \otimes \nu_T : \quad A^{\otimes S} \otimes \mathcal{C}_{T^{10}} \otimes A^{\otimes T} \otimes M \longrightarrow A^{\otimes S} \otimes M\ .$$

It follows from the algebra and module axioms that this is a $\mathcal{C}$-complex. Its realization $|\mathcal{K}|$ we denote usually by $\mathcal{K}_K(A\,; M)$, in order to stress that it depends in an essential way upon the ground ring K.

3.2 Γ-cotangent complex and Γ-homology groups. Let A be a subalgebra of the E_∞ algebra B, as in 3.1, and M a B-module, all these being assumed flat over K. We define the Γ-*cotangent complex* of B relative to A, with coefficients M to be the quotient

$$\mathcal{K}(B/A\,;M) \quad = \quad \mathcal{K}_K(B\,;M)/\mathcal{K}_K(A\,;M)\,.$$

(Here we have assumed that $A \subset B$ is a cofibration. In a more general situation the quotient can be replaced by the cofibre.) The Γ-*homology* of B over A is the homology of this complex:

$$H\Gamma_*(B/A\,;M) = H_*(\mathcal{K}(B/A\,;M))\,;$$

the Γ-*cohomology* is correspondingly defined by

$$H\Gamma^*(B/A\,;M) = H^*(\mathrm{RHom}_B(\mathcal{K}(B/A\,;B),M)\,.$$

3.3 Remark. For theoretical and practical reasons we have chosen to define the cotangent complex as a quotient. This avoids the need to handle derived tensor products over E_∞ algebras. It also has the pleasant consequence that the important transitivity theorem (3.4 below) becomes trivial to prove. This advantage is of course an illusion: the counterbalancing disadvantage is that in order for our definitions to be useful, we have to work to prove that $\mathcal{K}(B/A\,;M)$, and hence $H\Gamma_*(B/A\,;M)$, are essentially independent of the ground ring K. The proof of this flat base-change theorem occupies most of the rest of §3, and the Appendix.

3.4 Transitivity theorem. *Let $A \subset B \subset C$ be inclusions of E_∞-algebras, and M an E_∞ C-module. Then there is a cofibration sequence of* Γ-*cotangent complexes*

$$\mathcal{K}(B/A\,;M) \quad\to\quad \mathcal{K}(C/A\,;M) \quad\to\quad \mathcal{K}(C/B\,;M) \quad\to\quad S\mathcal{K}(B/A\,;M)$$

and therefore a long exact sequence of Γ-*homology groups*

$$\cdots \to H\Gamma_{n+1}(C/B\,;M) \to H\Gamma_n(B/A\,;M) \to H\Gamma_n(C/A\,;M) \to H\Gamma_n(C/B\,;M) \to \cdots$$

and similarly for cohomology.

Proof. The definitions require A, B and C to be replaced by corresponding projective resolutions. Then everything follows from the exact sequence connecting the cofibres of the three maps in the triple

$$\mathcal{K}_K(A\,;M) \to \mathcal{K}_K(B\,;M) \to \mathcal{K}_K(C\,;M)\,. \quad \square$$

We now begin on the definitions and lemmas which we shall need for the other main result of this section, the flat base-change theorem.

3.5 A model for the derived tensor product.
We propose that the derived tensor product of modules over an E_∞ algebra should be defined by the following construction. In Proposition 3.6 below, we shall justify it in the case of flat modules over a strictly commutative algebra, which is the only case we need in this paper.

Let $\mathcal{C}$ be a cofibrant cyclic E_∞ operad. We are going to make a realization like those in 2.8 and 2.9, but with $\mathbb{S}$ or $\mathbb{S}^1$ replaced by $\mathbb{S}^r$, which is the category of finite sets with r distinct basepoints $0_1, \ldots, 0_r$, and isomorphisms of sets which preserve the basepoints in order. There is formally no problem in extending the definition given in 2.8 to this case, except that when $S \sqcup T$ is a partition of the set $V = \hat{V} \sqcup \{0_1, \ldots, 0_r\}$ in $\mathbb{S}^r$, the structural map

$$\circ_{pq} : \mathcal{C}_{S^p} \otimes \mathcal{M}_{S \sqcup T} \to \mathcal{M}_{T^q}$$

must be zero when S contains more than one of the basepoints, as there is then no natural way to structure T^q as a space with r basepoints. (Here p and q are dummy labels.) When $r \geq 2$, the homology of the realization is much simpler than the homologies for which we obtained spectral sequences in 2.13, because the analogues of V_p are now free Σ_p-modules. That is why the following construction is valid.

Let A be a $\mathcal{C}$-algebra and let $M_1, \ldots, M_n$ be A-modules. Suppose that A and all the M_i are flat over the commutative ground ring K. We construct a $\mathcal{C}$-complex $\mathcal{M}$ on the category $\mathbb{S}^r$ by setting $\mathcal{M}_V = M_1 \otimes \cdots \otimes M_r \otimes A^{\otimes \hat{V}}$ when $V = \hat{V} \sqcup \{0_1, \ldots, 0_r\}$ is a set with r basepoints, and, when $V = S \sqcup T$, taking

$$\circ_{pq} : \mathcal{C}_{S^p} \otimes \mathcal{M}_{S \sqcup T} \to \mathcal{M}_{T^q}$$

to be

$$\circ_{pq} \quad = \quad \begin{cases} 1 \otimes \mu_S, & \text{if } S \text{ contains no basepoint} \\ 1 \otimes \nu^i_S, & \text{if } S \text{ contains } 0_i \text{ and no other} \\ 0, & \text{if } S \text{ contains more than one basepoint} \end{cases}$$

where μ is the structural map for the $\mathcal{C}$-algebra A, and ν^i is that for the module M_i.

We define the *left derived tensor product* $M_1 \overset{L}{\otimes}_A M_2 \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A M_r$ to be the realization $|\mathcal{M}|$.

The following proposition is sufficient justification, for the purposes of this paper, of the above definition. The proof actually goes rather further, as most of it does not need R-flatness.

3.6 Proposition. *Let R be a commutative algebra over the ground ring K, and let $M_1, \ldots, M_r$ (where $r \geq 2$) be complexes of R-modules, in the sense of standard homological algebra. Suppose that R is flat over K, and all the M_i are flat over R. Then there is a quasi-isomorphism*

$$M_1 \overset{L}{\otimes}_R M_2 \overset{L}{\otimes}_R \cdots \overset{L}{\otimes}_R M_r \quad \simeq \quad M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_r \ .$$

Proof. For transparency we treat first the case when $r = 2$. After 2.12, we may assume that $\mathcal{C}$ is the tree operad of 1.5. We shall also need the corresponding A_∞ operad, in which the $\mathcal{C}_{\mathrm{ord},\, n+2}$ is the chain complex of trees which can be embedded in the plane with labels $0_1, 1, 2, \ldots, n, 0_2$ in cyclic order. This operad has no permutations. It is well known that $\mathcal{C}_{\mathrm{ord}}$ is a subdivision of the Stasheff operad of associahedra, and that the homology of the nth complex modulo its boundary is a single copy of the ground ring. If we construct the realization $|\mathcal{M}|_{\mathrm{ord}}$ of the complex

$$\mathcal{M}_{0_1,1,2,\dots,n,0_2} \quad = \quad M_1 \otimes R^{\otimes n} \otimes M_2$$

with respect to $\mathcal{C}_{\mathrm{ord}}$, then the E^1-term of the skeletal spectral sequence is the bar resolution, and since everything is flat over K we have $E^2_{p,*} \approx \mathrm{Tor}^R_{p,*}(M_1, M_2)$. Examination of the attaching maps in the structure of $|\mathcal{M}|_{\mathrm{ord}}$ shows that $E^2 = E^\infty$ since R is strictly associative. Indeed we have assumed M_i is R-flat, so this is quite obvious, as $E^2_{p,*} \approx 0$ for $p > 0$, and the homology spectral sequence converges to $M_1 \otimes_R M_2$.

To complete the case $r = 2$ it now suffices to prove that the $\mathbb{S}^2$-realization $|\mathcal{M}|$, which we defined in 3.5, is equivalent to the ordered realization $|\mathcal{M}|_{\mathrm{ord}}$. There is certainly a natural map $|\mathcal{M}|_{\mathrm{ord}} \to |\mathcal{M}|$, induced by inclusion of operads. This map respects skeleta, so induces a map of homology spectral sequences. We have to calculate the E^1-term in the target spectral sequence. Now the homology modulo its boundary of the complex of trees with labels $\{0_1, 1, 2, \dots, n, 0_2\}$ is the tree representation, which restricts to the regular representation of Σ_n [17]; and the inclusion of the ordered trees induces a map which takes the homology generator to a generator of this regular module [19]. After taking Σ_n-covariants as the construction of $|\mathcal{M}|$ requires, we therefore have an isomorphism of E^2-terms. Thus the spectral sequences are isomorphic, and so $|\mathcal{M}|_{\mathrm{ord}} \to |\mathcal{M}|$ is a quasi-isomorphism. Combining this with the first result of the proof shows that $|\mathcal{M}|$ is quasi-isomorphic to $M_1 \otimes_R M_2$. The result is now proved for $r = 2$.

The proof for $r > 2$ follows exactly the same lines. The only difference is the inclusion of a counting argument to match the numbers of generators in the free modules involved in the two E^2-terms, for these no longer have rank one. We omit the details. $\square$

The following acyclicity lemma is central to the results of the present section.

3.7 Lemma. *Let K be a commutative ring, and M a K-module. Then the complex $\mathcal{K}_K(K\,;M)$ is acyclic.*

Contemplating configuration spaces makes one think that 3.7 should be true, but the only proof we know is combinatorial and lengthy. This proof is given in Appendix A. The first consequence of the Lemma is that one can calculate Γ-homology relative to the ground ring without normalizing by quotienting by $\mathcal{K}_K(K\,;M)$.

3.8 Proposition. *Let A be an E_∞ algebra over the ground ring K, and M an A-module. Then*

$$H\Gamma_*(A/K\,;M) \approx H_*(\mathcal{K}_K(A\,;M))\ .$$

Proof. Since quotienting by the acyclic complex $\mathcal{K}_K(K\,;M)$ is a quasi-isomorphism, we have $\mathcal{K}(A/K\,;M) \simeq \mathcal{K}_K(A\,;M)$. $\square$

Naturally, one would like to be able to describe $H\Gamma(B/A\,;M)$ in an equally simple way for any E_∞ pair of algebras $A \subset B$. The tensor powers of A would have to be replaced by derived tensor powers of B over A. We have little doubt that this could be done, but the resulting elegant statement might not justify the technical mischief with derived powers which would be needed to prove the result and, later, to apply it. The following theorem is a good substitute.

3.9 Theorem (Flat base-change for $\mathcal{K}(B/A;M)$). *Let K be a commutative ring, and R a flat commutative K-algebra. Then*

1. *For every E_∞ algebra A and every A-module M which are flat over the ground ring R, there is a quasi-isomorphism*
$$\mathcal{K}_R(A;M) \quad \simeq \quad \mathcal{K}_K(A;M)/\mathcal{K}_K(R;M)\,.$$
2. *If A is a subalgebra of the E_∞ algebra B and M is a B-module, all these being R-flat, then the quasi-isomorphism type of $\mathcal{K}(B/A;M)$ is the same, whether the ground ring be taken to be K or R.*

Proof. (1) Suppose L is a commutative ring such that $K \subset L \subset R$. In the application, L will actually be either K or R. As in 3.1, no generality is lost by assuming that R is a (strictly commutative) subalgebra of the E_∞ algebra A. We have defined $\mathcal{K}_L(A;M)$ as the realization of a certain $\mathcal{C}$-complex $\mathcal{K}$. This means that $|\mathcal{K}|'$ is defined first as a certain quotient of $\bigoplus_{n\geq 2}\mathcal{C}_{n+1}\otimes_{\Sigma_n} A^{\otimes n}\otimes M$, then $\mathcal{K}_L(A;M) = |\mathcal{K}|$ is constructed as the cofibre of a map $|\mathcal{K}|' \to A\otimes M$, all tensor products being over L.

We construct a filtration of $|\mathcal{K}|'$ and $A\otimes M$, and therefore of $\mathcal{K}_L(A;M)$, by defining the pth filtration stage $F^p\mathcal{K}_L(A;M)$ to be the image of the submodule in which at most p of the tensor factors from A lie outside R. This respects all necessary identifications, and is thus a valid definition of a filtration in which
$$\mathcal{K}_L(R;M) = F^0 \subset F^1 \subset \cdots \subset F^{p-1} \subset F^p \subset \cdots \subset F^\infty = \mathcal{K}_L(A;M)\,.$$
Let us consider the quotient F^p/F^{p-1}. Under the action of Σ_n, every tensor $a_1\otimes\cdots\otimes a_n\otimes m$ with p factors outside R is equivalent to an element in which only $a_1,\ldots,a_p$ are outside R; and modulo lower filtrations this element is unique up to the action of $\Sigma_p\times\Sigma_{n-p}$. Provided that R is L-flat, and $p\geq 1$, it follows from 3.8 that F^p/F^{p-1} is quasi-isomorphic to $E\Sigma_p\otimes_{\Sigma_p}(A/R)\otimes_R(A/R)\otimes_R\cdots\otimes_R(A/R)$, where there are p factors A/R. Now this is quite independent of L. So if we take the natural filtered map between the two filtered complexes
$$\mathcal{K}_K(A;M) \longrightarrow \mathcal{K}_R(A;M)$$
associated with the two choices $L = K$ and $L = M$, we know that it induces equivalences of filtration quotients F^p/F^{p-1} for all $p\geq 1$. Therefore the map
$$\mathcal{K}_K(A;M)/F^0\mathcal{K}_K(A;M) \longrightarrow \mathcal{K}_R(A;M)/F^0\mathcal{K}_R(A;M)$$
is a quasi-isomorphism. But $F^0\mathcal{K}_K(A;M) = \mathcal{K}_K(R;M)$, and $F^0\mathcal{K}_R(A;M) = \mathcal{K}_R(R;M)$ which is acyclic by 3.7, so this relation is precisely (1) of the statement.

(2) When A is a subalgebra of the E_∞ algebra B and M is a B-module, all these being R-flat, we have a diagram
$$\begin{array}{ccccc}
\mathcal{K}_K(R;M) & \xrightarrow{\approx} & \mathcal{K}_K(R;M) & & \\
\downarrow & & \downarrow & & \\
\mathcal{K}_K(A;M) & \longrightarrow & \mathcal{K}_K(B;M) & \longrightarrow & \mathcal{K}_1(B/A;M) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_R(A;M) & \longrightarrow & \mathcal{K}_R(B;M) & \longrightarrow & \mathcal{K}_2(B/A;M)
\end{array}$$
in which two columns are cofibrations by (1) above, and two rows are cofibrations by definition. The diagram implies that the vertical map on the right between the two models for $\mathcal{K}(B/A;M)$ is a quasi-isomorphism. $\square$

3.10 Cyclic Γ-homology and cohomology. Let A be an algebra over the cofibrant cyclic E_∞ operad $\mathcal{C}$, with K as ground ring. We then have the cyclic $\mathcal{C}$-complex $|\mathcal{M}|$ described in 2.5, which has $\mathcal{M}_S = A^{\otimes S}$, with structural maps induced by the multiplication in A. We denote the cyclic realization of $|\mathcal{M}|$ by $\mathcal{K}^{\mathrm{cy}}(A)$.

The *cyclic* Γ*-homology* and *cyclic* Γ*-cohomology* are defined in terms of the cyclic realization $\mathcal{K}^{\mathrm{cy}}(A)$:

$$H\Gamma^{\mathrm{cy}}_*(A) = H_*(\mathcal{K}^{\mathrm{cy}}(A))$$
$$H\Gamma_{\mathrm{cy}}^*(A) = H^*(\mathrm{Hom}_K(\mathcal{K}^{\mathrm{cy}}(A), K))\ .$$

4. The A_∞ analogue: Hochschild and cyclic homology

The above theory is specifically for E_∞ structures, and is new. We now construct the precise analogue for A_∞ (homotopy-associative) structures, and show that this just leads to a new description of the familiar Hochschild homology and cyclic homology of associative algebras.

We replace $\mathbb{S}$ with the category $\mathbb{S}^{\mathrm{cy}}$ of *cyclically-ordered* finite sets and order-preserving isomorphisms. The automorphism group of an object of $\mathbb{S}^{\mathrm{cy}}$ is a finite cyclic group. If 0 is chosen as basepoint in an object S^0 of $\mathbb{S}$, its complement S is totally ordered, and the group of automorphisms preserving the basepoint is trivial.

We redefine operads and cyclic operads for the new case, replacing the category $\mathbb{S}$ in 1.1 and 1.2 by $\mathbb{S}^{\mathrm{cy}}$. The composition operations have the form

$$\circ_{s,t} : \mathcal{A}_S \otimes \mathcal{A}_T \to \mathcal{A}_{S \sqcup_{s,t} T}$$

where $S \sqcup_{s,t} T$ has the unique cyclic ordering obtained by concatenating the total orderings on $S\backslash\{s\}$ and $T\backslash\{t\}$. We say a cyclic operad $\mathcal{A}$ is A_∞ if $\mathcal{A}_S$ is contractible for each S, and the cyclic group C_S acts freely on $\mathcal{A}_S$. Cofibrancy is defined as before. Next we introduce algebras over a cyclic A_∞ operad $\mathcal{A}$, and modules over these algebras by analogy with 2.1. The simplest examples are associative rings and bimodules respectively. Similarly, cyclic and non-cyclic $\mathcal{A}$-complexes are defined by precise analogy with 2.4 and 2.6. The archetypes are $\mathcal{M}_S = A^{\otimes S}$ in the cyclic case, and $\mathcal{M}_{S^0} = A^{\otimes S} \otimes M$ in the non-cyclic case, where A is an associative or A_∞ algebra and M an A-bimodule. The realizations $|\mathcal{M}|$ and $|\mathcal{M}|_{\mathrm{cy}}$ are defined just as in 2.8 and 2.9, the category $\mathbb{S}$ being replaced everywhere by $\mathbb{S}^{\mathrm{cy}}$ and the symmetric group Σ_{n+1} in 2.9 by the cyclic group C_{n+1}.

Homology of the A_∞ realization.

We have the following analogue of 2.13. It is very much simpler than the E_∞ version, because the represention V_p is replaced the homology of the space of cyclically-ordered p-trees, which is free of rank one.

4.1 Proposition.

(1) *Let $\mathcal{M}$ be a $\mathcal{A}$-complex, where $\mathcal{A}$ is an A_∞ operad. Then there is a homology spectral sequence*

$$E^1_{p-1,q} \approx H_q(\mathcal{M}_{p+1}) \Longrightarrow H_{p+q-1}(|\mathcal{M}|)\ .$$

(2) *When $\mathcal{M}$ is a cyclic $\mathcal{A}$-complex the spectral sequence has the form*

$$E^1_{p-1,q} \approx H_q(EC_{p+1} \otimes_{C_{p+1}} \tilde{\mathcal{M}}_{p+1}) \Longrightarrow H_{p+q-1}(|\mathcal{M}|_{\mathrm{cy}})$$

where $\tilde{\mathcal{M}}_{p+1}$ indicates that the C_{p+1}-module structure of $\mathcal{M}_{p+1}$ is twisted by the sign representation.

Proof. Just as in the E_∞ case of 2.12, the spectral sequence obtained from the skeletal filtration of $|\mathcal{M}|$ or $|\mathcal{M}_{\mathrm{cy}}|$ is independent of the particular cofibrant cyclic A_∞ operad used in the construction. We may therefore choose the A_∞ *tree operad* $\mathcal{T}^{\mathrm{cy}}$, which is constructed just as in 1.5, but with the category $\mathbb{S}$ replaced by the category $\mathbb{S}^{\mathrm{cy}}$ of cyclically-ordered sets. This leads to the E^1 terms given above. □

4.2 Corollary.

(1) *Let $\mathcal{M}$ be the $\mathcal{A}$-complex with $\mathcal{M}_{S^0} = A^{\otimes S} \otimes M$, where A is an associative K-algebra and M an A-bimodule. Then the homology of $|\mathcal{M}|$ is the Hochschild homology of A, with dimension shifted by one:*

$$H_r(|\mathcal{M}|) \approx HH_{r+1}(A; M) \qquad \text{for } r \geq 0.$$

(2) *Let $\mathcal{N}$ be the cyclic $\mathcal{A}$-complex with $\mathcal{N}_S = A^{\otimes S}$. Then the homology of $|\mathcal{N}|_{\mathrm{cy}}$ is the cyclic homology of A, with a dimension shift:*

$$H_r(|\mathcal{N}|_{\mathrm{cy}}) \approx HC_{r+1}(A) \qquad \text{for } r \geq 0.$$

Proof. (1) Since $\mathcal{M}_p$ is discrete, the E^1 term of the spectral sequence of 4.1(1) collapses to the edge $E^1_{p-1,0} = A^{\otimes p} \otimes M$. Analysis of the identifications in $|\mathcal{M}|$ shows that $d^1_{p-1,0} : A^{\otimes p} \otimes M \to A^{\otimes(p-1)} \otimes M$ is the Hochschild boundary. Thus $E^1_{*,0}$ is simply the standard Hochschild complex, shifted down and truncated.

(2) Using a model where $\mathbb{S}^{\mathrm{cy}}$ has one set of each size, we have

$$\mathcal{T}^{\mathrm{cy}}_n \approx C_*(C_{n+1}) \otimes C_*(\tilde{T}^{\mathrm{cy}}_n)$$

where $\tilde{T}^{\mathrm{cy}}_n$ is the space of planar n-trees, and $C_*(C_{n+1})$ is the bar construction on the cyclic group which permutes the labels $\{0, 1, \ldots, n\}$ of these trees. Therefore $|\mathcal{N}|_{\mathrm{cy}}$ is a bicomplex which has $(m, k+1)$st group

$$\bigoplus_n C_m(C_{n+1}) \otimes C_k(\tilde{T}^{\mathrm{cy}}_n, T^{\mathrm{cy}}_n) \otimes A^{\otimes(n+1)}$$

where T^{cy}_n is the boundary of $\tilde{T}^{\mathrm{cy}}_n$ (the fully-grown trees). We filter by n. Since the complex $\tilde{T}^{\mathrm{cy}}_n$ is a Stasheff $(n-2)$-cell, $C_*(\tilde{T}^{\mathrm{cy}}_n, T^{\mathrm{cy}}_n)$ has only one homology group, generated by the homology class $[c_n]$ of the cycle denoted c_n in [19]. Thus each filtration quotient is a bicomplex for which the second standard spectral sequence (column homology first) collapses. We conclude that the spectral sequence associated to our filtration has $E^1_{n,m-1} \approx H_m(C_{n+1}; A^{\otimes(n+1)})$, where the action of the cyclic group on the tensor product includes the usual sign.

On the other hand, the cyclic homology of A is given by Tsygan's bicomplex. This has $A^{\otimes(n+1)}$ in the (m, n)th position, and the horizontal differentials are alternately T and N, the morphisms in the standard perodic resolution of the cyclic group C_{n+1}. Filtration by n gives rise to a spectral sequence with $E^1_{n,m} \approx H_m(C_{n+1}; A^{\otimes(n+1)})$.

There is an equivalence from the periodic resolution to the bar resolution which takes the generator to $[N|T|\ldots|T|N|T]$ in even degrees, and to $[T|N|\ldots|T|N|T]$ in odd degrees. We use it to construct a chain map θ from Tsygan's bicomplex (with the row $n = 0$ deleted) to the bicomplex representing $|\mathcal{N}|_{\mathrm{cy}}$. Explicitly, we define

$$\theta_{m,n} : A^{\otimes n+1} \longrightarrow C_m(C_{n+1}) \otimes C_{n-2}(\tilde{T}_n^{\mathrm{cy}}, T_n^{\mathrm{cy}}) \otimes A^{\otimes n+1}$$

by setting

$$\theta_{m,n}(a) = \begin{cases} [N|T| \dots |N|T] \otimes c_n \otimes a, & \text{for } m \text{ even} \\ [T|N| \dots |N|T] \otimes c_n \otimes a, & \text{for } m \text{ odd.} \end{cases}$$

The map θ commutes with horizontal differentials, since we began with a map of C_{n+1}-complexes. To prove that it commutes with vertical differentials, one needs a calculation like that which proves that Tsygan's diagram is a bicomplex, and the fact that the vertical Hochschild differential (arising from the identifications in $|\mathcal{N}|_{\mathrm{cy}}$) carries c_n to c_{n-1}. Finally, θ is a map of filtered bicomplexes which has bidegree $(0, -1)$ and which induces an isomorphism on the E^1 terms of the associated spectral sequences. Hence θ induces isomorphisms $HC_{r+1}(A) \approx H_r(|\mathcal{N}|_{\mathrm{cy}})$. □

5. Explicit complexes in the strictly commutative case

Let B be a strictly commutative algebra which is flat over a commutative ring A and let M be a B-module. By 3.8 and 3.9, we may take A as the ground ring in calculating $\mathcal{K}(B/A\,; M)$ and $H\Gamma_*(B/A\,; M)$. Accordingly we denote $\otimes_A$ simply by $\otimes$. The Γ-cotangent complex $\mathcal{K}(B/A\,; M)$ is quasi-isomorphic to $\mathcal{K}_A(B\,; M)$ by 3.9. When constructed using the tree operad $\mathcal{T}$, this is a bicomplex

$$C\Gamma_{p,q}(B/A\,; M) = \big(C_{q+2}(\mathbb{S}^1) \otimes_{\mathbb{S}^1} C_{p-1}(\tilde{T}_\bullet, T_\bullet)\big) \otimes_{\mathbb{S}^1} (B^{\otimes\circ} \otimes M)\,. \tag{5.1}$$

Here $\bullet$ denotes a generic object of $\mathbb{S}^1$, and $\circ$ denotes the same object minus its basepoint. The vertical differential d'' of the bicomplex is the differential of the two-sided bar construction on the category $\mathbb{S}^1$. The horizontal differential d' is the differential in the chain complex $C_*(\tilde{T}_\bullet)$, except that chains in the boundary $T_\bullet$ are identified with lower skeleta by relation 2.8(2). (When $n = 1$, the relative chain complex $C_*(\tilde{T}_n, T_n)$ has to be interpreted conventionally as A in degree -1.)

We can make this smaller and more explicit by replacing $\mathbb{S}^1$ with the model in which there is just one object $\{0, 1, \dots, k\}$ for each $k \geq 1$. Then one has to make many choices about how to identify an arbitrary quotient set of $\{0, 1, \dots, k\}$ with some $\{0, 1, \dots, l\}$. (See, for example, the labelling convention described in the Appendix.) Any coherent system of choices gives a complex

$$C\Gamma_{p,q}(B/A\,; M) = \bigoplus_{k\geq 1} \big(C_{q+2}(\Sigma_k) \otimes_{\Sigma_k} C_{p-1}(\tilde{T}_k, T_k)\big) \otimes_{\Sigma_k} B^{\otimes k} \otimes M$$

which is quasi-isomorphic to (5.1), though the precise horizontal differential d' depends upon the choices. Once more, the vertical differential is that of the two-sided bar construction on the symmetric groups Σ_k. There is a dual version for cohomology when B is projective.

Since we are working in the discrete case, the subsidiary spectral sequence of 2.13 collapses to an edge and we have the following spectral sequence.

$$E^1_{p-1,q} \approx H_q(\Sigma_p\,; V_p \otimes B^{\otimes p} \otimes M) \Longrightarrow H\Gamma_{p+q-1}(B/A\,; M) \tag{5.2}$$

where V_p is the Σ_p-module given by the reduced homology of the tree-space T_p. When B is projective, there is a dual spectral sequence in cohomology

$$E_1^{p-1,q} \approx H^q(\Sigma_p\,; V_p \otimes \mathrm{Hom}(B^{\otimes p}, M)) \Longrightarrow H\Gamma^{p+q-1}(B/A\,; M)\,. \tag{5.3}$$

5.4 Theorem [19]. *The edge $q = 0$ of the spectral sequence above is precisely the complex used in defining the Harrison (co)homology* [12] $\mathrm{Harr}_*(B/A\,;M)$ *of* B *(with a shift in degree). Therefore there are natural transformations*

$$H\Gamma_{p-1}(B/A\,;M) \to \mathrm{Harr}_p(B/A\,;M)\,, \qquad H\Gamma^{p-1}(B/A\,;M) \leftarrow \mathrm{Harr}^p(B/A\,;M)$$

when B *is flat (resp. projective), which are isomorphisms when* B *contains a field of characteristic zero.*

Proof. We give the details for homology. The edge of the spectral sequence (5.2) has terms:

$$E^1_{p-1,0} \;\approx\; H_0(\Sigma_p\,;V_p \otimes B^{\otimes p} \otimes M) \;\approx\; V_p \otimes_{\Sigma_p} B^{\otimes p} \otimes M\ .$$

Now we describe the structure of the Σ_p-module $V_p = H_{p-3}(T_p)$; further details can be found in [17], [19]. The tree space T_p has the homotopy type of a wedge of $(p-1)!$ spheres of dimension $p-3$. A set of independent homology generators is given by $\{\pi c_p \,|\, \pi \in \Sigma_{p-1}\}$, where c_p is the cycle consisting of cyclically labelled trees in the plane. Let $s_{i,p-i} = \sum \varepsilon_\pi \pi^{-1}$, where ε_π is the sign of π, and the sum is over $(i, p-i)$-shuffles in Σ_p. In [19] it is shown that $s_{i,p-i}c_p = 0$ for $i = 1, \ldots, p-1$ and that these relations completely determine the Σ_p-module structure of V_p. It follows that $V_p \otimes_{\Sigma_p} B^{\otimes p}$ is isomorphic to $B^{\otimes p}$ modulo the submodule of shuffle decomposables.

It remains to identify the differential $d^1 : E^1_{p,0} \to E^1_{p-1,0}$. It is straightforward to check that $d^1(c_p \otimes x_1 \otimes \cdots \otimes x_p \otimes m) = c_{p-1} \otimes b(x_1 \otimes \cdots \otimes x_p \otimes m)$, where b denotes the usual Hochschild boundary map. The edge $E^1_{*,0}$ is therefore the quotient of the Hochschild complex by the shuffle decomposables, which is precisely Harrison's complex. Hence, $E^2_{p-1,0} \approx \mathrm{Harr}_p(B/A\,;M)$.

The edge map of the spectral sequence gives a natural transformation

$$H\Gamma_{p-1}(B/A\,;M) \to \mathrm{Harr}_p(B/A\,;M)\ .$$

When B contains a field of characteristic zero, the higher homology of the symmetric groups is zero, so the spectral sequence collapses to the edge and the above is an isomorphism. $\square$

5.5 Proposition.

(1) $\quad H\Gamma_0(B/A\,;M) \approx \Omega_{B/A} \otimes_B M\ , \qquad H\Gamma^0(B/A\,;M) \approx \mathrm{Der}_A(B,M)\ ;$

(2) $\quad H\Gamma^1(B/A\,;M) \approx \mathrm{Exalcom}_A(B,M)\ .$

(Here $\mathrm{Exalcom}_A(B,M)$, the module of infinitesimal A-algebra extensions of B by M, is as defined in [11], 0_{IV} §18.)

Proof. (1) In the bicomplex (5.1), $C\Gamma_{0,0}/d''(C\Gamma_{0,1}) \approx B \otimes M$. The image of the horizontal differential $d' : C\Gamma_{1,0} \to C\Gamma_{0,0}$ is spanned by the usual relations for differentials of products. It follows that $H\Gamma_0(B/A\,;M)$ is the module of Kähler differentials $\Omega_{B/A} \otimes_B M$. Similarly, the zeroth cohomology group is $\mathrm{Der}_A(B,M)$.

(2) Suppose first that B is A-projective. In the spectral sequence (5.3) we have $E_2^{0,1} \approx 0$, $E_2^{1,0} \approx \mathrm{Harr}^2(B/A;M)$ and so $H\Gamma^1(B/A;M) \approx \mathrm{Harr}^2(B/A;M)$. This is the module of A-split infinitesimal commutative A-algebra extensions of B by

M. Since B is projective this coincides with $\mathrm{Exalcom}_A(B, M)$, the module of all infinitesimal A-algebra extensions of B by M.

In the general case when B is not A-projective we have to use a simplicial resolution and elementary properties of André/Quillen cohomology. It is elementary that Γ-(co)homology extends to simplicial rings, with coefficients in a simplicial module: the Γ-cotangent complex (3.2) of a simplicial ring is a simplicial dg-module, and one simply takes the associated total complex. The cofibrancy of the operad ensures that this is a homotopy invariant of the simplicial ring. This said, we may replace the algebra B by an André/Quillen resolution P_* consisting of polynomial algebras over A. Filtering the Γ-cotangent complex by the simplicial degree gives a spectral sequence

$$E_1^{p,q} = H\Gamma^q(P_p/A\,; M) \Longrightarrow H\Gamma^{p+q}(B/A\,; M)\ .$$

On the edge we have $E_1^{p,0} = H\Gamma^0(P_p/A\,; M) = \mathrm{Der}_A(P_p, M)$ by (1), so by definition the André/Quillen cohomology $D^p(B/A\,; M)$ is just $E_2^{p,0}$. In particular, $E_2^{1,0} \approx \mathrm{Exalcom}_A(B, M)$. But $E_1^{p,1} = 0$ by the first case, since P_p is projective and polynomial. The spectral sequence now gives the result. □

5.6 Corollary. *When B contains a field of characteristic zero,*

$$H\Gamma_p(B/A\,; M) \approx D_p(B/A\,; M)\ , \qquad H\Gamma^p(B/A\,; M) \approx D^p(B/A\,; M)\ ,$$

where D_* is André/Quillen homology.

Proof. Again we give the details for homology. If B is flat over A and contains a field of characteristic zero then Harrison homology coincides with André/Quillen homology [14] so the result is given by 5.4. If B is not flat, we replace it by a simplicial André resolution by polynomial algebras, P. (As in the proof of 5.5 this is the preferred method for strictly commutative rings.) We again obtain a spectral sequence: $E^1_{p,q} = H\Gamma_q(P_p/A\,; M) \Longrightarrow H\Gamma_{p+q}(B/A; M)$. Since each P_i is flat, 5.3 gives $H\Gamma_0(P_i/A; M) = \Omega_{P_i/A} \otimes_{P_i} M$, and all higher homology groups are zero by 5.4. Thus the spectral sequence collapses to the edge, where $E^1_{*,0}$ is exactly an André/Quillen resolution of B, giving the result. The case of cohomology is similar, except that 'flat' is everywhere replaced by 'projective'. □

In general Γ-homology is different from André/Quillen homology and from Harrison homology. The following example shows this, and reveals a non-trivial differential in the spectral sequence of 5.2.

5.7 Example. First take $B = A = \mathbb{F}_2$. Then $1 \otimes 1 \otimes 1 \otimes 1$ is a non-bounding Harrison 4-cycle, by the calculation in ([2], §4). Thus $\mathrm{Harr}_4(\mathbb{F}_2/\mathbb{F}_2; \mathbb{F}_2) \not\approx 0$, and by 5.4 our element $1 \otimes 1 \otimes 1 \otimes 1$ exists in $E^2_{3,0}$. Since 3.7 or the transitivity theorem 3.4 implies that $H\Gamma_3(\mathbb{F}_2/\mathbb{F}_2; \mathbb{F}_2) \approx 0$ (and similarly for André homology), this cycle must map by the only available differential d^2 to a non-zero element of $E^2_{1,1}$. (The only such element is $\alpha \otimes 1 \otimes 1$, where α generates $H_1(\Sigma_2; \mathbb{F}_2)$; for the module V_2 is trivial). Thus $H\Gamma_3 \not\approx \mathrm{Harr}_4$.

Now let us take B to be the polynomial algebra $\mathbb{F}_2[X]$, $A = \mathbb{F}_2$, $M = B/(X) \approx \mathbb{F}_2$. A brief calculation with shuffles shows that $E^2_{3,0} \approx \mathrm{Harr}_4(\mathbb{F}_2[X]/\mathbb{F}_2; \mathbb{F}_2)$ contains no non-zero element of degree two in X. Therefore $\alpha \otimes X \otimes X \in E^2_{1,1}$ is an infinite cycle which is not in the image of d^2 and therefore is not a boundary. So $H\Gamma_2(\mathbb{F}_2[X]/\mathbb{F}_2; \mathbb{F}_2) \not\approx 0$, and $H\Gamma_2$ is not André's H_2.

5.8 Theorem.

(1) *Let B and C be A-algebras, with B flat over A, and let M be a $B \otimes_A C$-module. Then the complex $\mathcal{K}(B \otimes_A C/C\,; M)$ is quasi-isomorphic to $\mathcal{K}(B/A\,; M)$, so that*

$$H\Gamma_*(B \otimes_A C/C\,; M) \quad \approx \quad H\Gamma_*(B/A\,; M)\ .$$

(2) *Let B and C be flat A-modules, and M a $B \otimes_A C$-module. Then there is a quasi-isomorphism*

$$\mathcal{K}(B \otimes_A C/A\,; M) \quad \simeq \quad \mathcal{K}(B/A\,; M) \quad \oplus \quad \mathcal{K}(C/A\,; M)\ ,$$

and therefore $H\Gamma_*(B \otimes_A C/A\,; M) \approx H\Gamma_*(B/A\,; M) \oplus H\Gamma_*(C/A\,; M)$.

(3) *If B is an étale A-algebra, then $H\Gamma_*(B/A\,; M) \approx H\Gamma^*(B/A\,; M) \approx 0$ for every B-module M.*

Proof. (1) Since B is flat over the discrete commutative ring A, the cotangent complex $\mathcal{K}(B/A\,; M)$ is equivalent to $\mathcal{K}_A(B\,; M)$. Also $B \otimes_A C$ is flat over C, and $\mathcal{K}(B \otimes_A C; M)$ may be replaced by $\mathcal{K}_C(B \otimes_A C\,; M)$. But standard identities with the tensor product show that $\mathcal{K}_C(B \otimes_A C\,; M) \approx \mathcal{K}_A(B\,; M)$, because these are realizations of isomorphic complexes.

(2) We have an exact triangle corresponding to the triple $A \to C \to B \otimes_A C$

$$\mathcal{K}(C/A\,; M) \to \mathcal{K}(B \otimes_A C/A\,; M) \to \mathcal{K}(B \otimes_A C/C\,; M)\ .$$

Using the quasi-isomorphism of (1), this can be split by the map

$$\mathcal{K}(B/A\,; M) \to \mathcal{K}(B \otimes_A C/A\,; M)\ .$$

(3) The arguments of André ([1], §20), for the homology of a separable field extension generalize to show that this can be deduced from (1), (2) and the long exact sequence of a triple, as was observed by Quillen ([14], §5). □

6. A product

In this section we prove the following theorem, giving a graded anti-commutative product in the Γ-cohomology of a commutative algebra. This product is not associative. We believe it is a graded Lie product, but we have not yet verified all the details of the Jacobi identity.

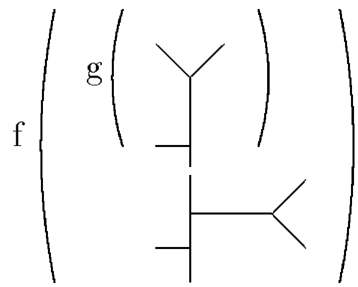
6.1 Theorem. *There is a graded anti-commutative product in Γ-cohomology*

$$[-,-] : H\Gamma^l(B/A\,; B) \otimes H\Gamma^m(B/A\,; B) \to H\Gamma^{l+m}(B/A\,; B)\ .$$

We begin by explaining the idea of the construction, which mimics the Lie bracket in Hochschild cohomology [6]. We recall that this is defined as a graded commutator of circle products, where the circle product $f \circ g$ is an alternating sum over i of 'substitution of g into f in the i-th place'. As in §5, realization using the tree operad gives rise to the following bicomplex for Γ-cohomology of a discrete commutative algebra B, in which (as before) $\circ$ denotes the complement of the basepoint in the set $\bullet$ of the category $\mathbb{S}^1$:

$$\begin{aligned} C\Gamma^{p,q}(B/A\,; B) &= \operatorname{Hom}\Big(\big(C_{q+2}(\mathbb{S}^1) \otimes_{\mathbb{S}^1} C_{p-1}(\tilde{T}_\bullet, T_\bullet)\big) \otimes_{\mathbb{S}^1} B^{\otimes \circ}, B\Big) \\ &\approx \operatorname{Hom}_{\mathbb{S}^1}\Big(C_{q+2}(\mathbb{S}^1) \otimes_{\mathbb{S}^1} \tilde{C}_{p-2}(T_\bullet), \operatorname{Hom}(B^{\otimes \circ}, B)\Big)\ . \end{aligned}$$

For cochains $f \in C\Gamma^l$, $g \in C\Gamma^m$, the bracket $[f,g]$ is defined using a difference $f \circ g - (-1)^{lm} g \circ f$ where $f \circ g \in C\Gamma^{l+m}$ is the sum over all possible ways of 'inserting g into f'; as indicated schematically by the diagram



However, it is necessary to use a diagonal approximation in the construction. Lack of strict commutativity complicates matters and forces us to add a correction term to our bracket.

Now we give the details of the proof of Theorem 6.1. The first ingredient is the following co-operad structure (which is closely related to a co-operad discussed by Ginzburg and Kapranov ([10], §3.5)).

6.2 Lemma. *The chain complexes* $\{\widetilde{C}_*(T_{U^0})[-2]\,;\, U^0 \in \mathbb{S}^1\}$ *form a co-operad.*

Proof. We define $\theta_{V,W} : \widetilde{C}_{*-2}(T_{U^0}) \to \widetilde{C}_{*-2}(T_{V^{01}}) \otimes \widetilde{C}_{*-2}(T_{W^0})$, for $U^0 = V^0 \sqcup W$. An internal edge in a U^0-tree t divides the tree into two parts. If t has an internal edge such that one of these parts is labelled by V^0 and the other by W, then $\theta_{V,W}(t)$ is given by cutting t at this internal edge to produce a tree labelled by V^{01} and a tree labelled by W^0. If t has no such internal edge we set $\theta_{V,W}(t) = 0$. (The new labelling sets V^{01}, W^0 are best thought of as quotient sets of U^0 obtained by identifying all elements of W, V^0 respectively.) It is easy to see that the $\theta_{V,W}$'s are chain maps, satisfying the required co-associativity condition. □

Secondly we need a diagonal approximation on the chains on the category $\mathbb{S}^1$. Recall that such a diagonal approximation Δ exists and that for the bar resolution it may be chosen to be strictly coassociative and cocommutative up to homotopy, $\Delta \simeq \tau\Delta$. The homotopy, H say, is itself commutative up to homotopy. Now we combine the diagonal approximation Δ with taking induced isomorphisms on quotient sets. For each partition $U^0 = V^0 \sqcup W$ we have a chain map $\phi^{\Delta}_{V,W}$: $C_*(\mathbb{S}^1) \to C_*(\mathbb{S}^1) \otimes C_*(\mathbb{S}^1)$,

$$[\varphi_1|\ldots|\varphi_k] \mapsto \sum_i [\check{\varphi}_1|\ldots|\check{\varphi}_i]\check{\varphi}_{i+1}\ldots\check{\varphi}_k \otimes \hat{\varphi}_1\ldots\hat{\varphi}_i[\hat{\varphi}_{i+1}|\ldots|\hat{\varphi}_k]\,,$$

where the φ's start at U^0, the $\check{\varphi}$'s at V^{01} and the $\hat{\varphi}$'s at W^0. We denote by ϕ^H such maps constructed with the homotopy H in place of Δ and so on.

Finally, we have the structure maps of the endomorphism operad of B,

$$\psi_s : \mathrm{Hom}(B^{\otimes S}, B) \otimes \mathrm{Hom}(B^{\otimes T}, B) \to \mathrm{Hom}(B^{\otimes S \sqcup_s T}, B)\,,$$

for each element s of S.

Now the map $- \circ - : C\Gamma^l \otimes C\Gamma^m \to C\Gamma^{l+m}$ is given by

$$f \circ g = \prod_{U^0 \in \mathbb{S}^1} \sum_{U^0 = V^0 \sqcup W} \psi_{\bar{1}}(f \otimes g)(1 \otimes \tau \otimes 1)(\phi^{\Delta}_{V,W} \otimes \theta_{V,W})\,,$$

where τ is a suitably signed switch of factors and $\tilde{1}$ is the image under $\check{\varphi}_1 \dots \check{\varphi}_k$ of $1 \in V^{01}$.

Now consider the graded commutator

$$\langle f, g\rangle = f \circ g - (-1)^{lm} g \circ f\,.$$

We check how this behaves with respect to the differentials. Since the $\phi^{\Delta}_{V,W}$'s are chain maps, we have $d''(f \circ g) = (-1)^m d'' f \circ g + f \circ d'' g$, for the vertical differential d''. The horizontal differential, d', consists of the internal boundary in the tree spaces δ plus extra terms, $d'' = \delta + D$ say. Again $\delta(f \circ g) = (-1)^m \delta f \circ g + f \circ \delta g$, since the $\theta_{V,W}$'s are chain maps. Analysis of the identifications in the cotangent complex shows that D can be expressed in terms of the circle product. Since B is strictly commutative, we may consider the product cochain $\rho \in C\Gamma^1$; that is $\rho(\Gamma_{S^0})(\otimes_{s\in S} b_s) = \prod_{s \in S} b_s$, where Γ_S is the star tree labelled by S^0. Then $Df = f \circ \rho - (-1)^l \rho \circ f$. A calculation shows

$$D\langle f, g\rangle = \langle Df, g\rangle + \langle f, Dg\rangle + E(f,g)\,,$$

where $E(f,g) \in C\Gamma^{l+m+1}$ is an error term which results from the diagonal approximation not being strictly commutative. It can be described as follows.

For $U^0 = X^0 \sqcup Y \sqcup Z$ we define $\overline{\theta}_{X,Y,Z} : \widetilde{C}_p(T_{U^0}) \to \widetilde{C}_{-1}(T_{X^{012}}) \otimes \widetilde{C}_{p'}(T_{Y^0}) \otimes \widetilde{C}_{p-p'-1}(T_{Z^0})$, as follows. Suppose a U^0-tree t has exactly two internal edges meeting at the root, the part above one being a subtree labelled by Y and the part above the other being a subtree labelled by Z. Then $\overline{\theta}_{X,Y,Z}(t)$ is given by splitting the tree t in the evident manner at these internal edges into a star tree labelled X^{012} and two subtrees labelled Y^0 and Z^0. If the tree t cannot be split as indicated we set $\overline{\theta}_{X,Y,Z}(t) = 0$.

We have

$$E(f,g) = \prod_{U^0 \in \mathbb{S}^1} \sum_{U^0 = X^0 \sqcup Y \sqcup Z} \psi_2(\psi_1 \otimes 1)(\rho \otimes f \otimes g)(1243)(\phi^{\Delta - \tau\Delta}_{Y,Z} \otimes \overline{\theta}_{X,Y,Z})\,,$$

where the permutation (1243) is simply the necessary reordering of factors (with appropriate sign). Now define $f \bullet g \in C\Gamma^{l+m}$ by the same formula as for $E(f,g)$, but using the homotopy H to replace $\phi^{\Delta-\tau\Delta}$ by ϕ^H. Then, by construction, we have $d''(f \bullet g) = (-1)^m d'' f \bullet g + f \bullet d'' g + E(f,g)$ and by considering the maps $\bar{\theta}$ it is not hard to check that $d'(f \bullet g) = (-1)^m d' f \bullet g + f \bullet d' g$.

Finally, define a bracket by

$$[f,g] = \langle f, g\rangle + f \bullet g\,.$$

From the discussion above, this map is well-behaved with respect to the differentials and so induces a map in cohomology.

Using the fact that H is commutative up to a homotopy, H' say, we may construct a homotopy between $[f,g]$ and $-(-1)^{lm}[g,f]$. So the bracket is graded anti-commutative in cohomology. This completes the proof of Theorem 6.1. □

Remarks. The bracket described above is compatible with the Lie product in Harrison cohomology ([7], §5.7). For 0-cocycles it is simply the usual bracket of derivations.

If n is odd or the characteristic of B is 2, then the circle product $g \mapsto g \circ g$ passes to cohomology giving an operation $H\Gamma^n(B/A, B) \to H\Gamma^{2n}(B/A, B)$.

If the same constructions are carried out in the A_∞ situation of §4, the error term $E(f, g)$ is always zero and one recovers the Lie product of Gerstenhaber on Hochschild cohomology ([6], §7).

Appendix A: Acyclicity of $\mathcal{K}_A(A; M)$

Contraction of a certain complex without permutations

We construct, then contract, a certain chain complex related to $\mathcal{K}_A(A; M)$. It is obtained by glueing together the chains on the various tree spaces $\tilde{T}_n$, for $n \geq 2$. For simplicity we may as well take M to be the ground ring A. The construction of our complex K_∞ requires a **labelling convention** for trees, which is detailed below. The contraction requires an **ordering convention** for the edges of a tree. Both these conventions are somewhat arbitrary at this stage, but they have to be compatible with each other.

Ordering convention. Let $t \in \tilde{T}_n$ be an n-tree. It therefore has a root labelled 0, and leaves labelled $1, 2, \ldots, n$. Let β_i be the arc (shortest path) in t from the leaf i to the root. Then $t = \bigcup_{i=0}^{n} \beta_i$. We introduce a total ordering on the set of edges of t as follows. If x, y are edges, then x precedes y (written $x < y$) if **either** x and y are in some common arc β_i with y nearer the root, **or** $\min\{i \mid x \in \beta_i\} > \min\{j \mid y \in \beta_j\}$. This does define a total ordering, in which an internal edge occurs at the first moment after all edges above it have been counted. When no internal edge is available, the next leaf (in descending order) is taken. So the leaf n is, perversely, first. The root is last.

The trees t/x and $t\backslash x$. An internal edge x in an n-tree t divides the tree into two. The portion including the root (and the edge x itself) is a sub-tree called $t\backslash x$. The other part, containing some leaves and x itself but not the root, is called the sub-tree over x and is written t/x. It is much better to regard t/x as the identification space obtained by crushing the sub-tree $t\backslash x$ to a single edge, and $t\backslash x$ as obtained by identifying t/x to an edge. (If x is a leaf or the root of t, the symbols t/x and $t\backslash x$ are interpreted as either the whole of t or the tree consisting of a single leaf, as appropriate.) Now we have to decide how to label these quotient trees.

Labelling convention. A quotient tree such as t/x is naturally labelled by *subsets* forming a partition of the set $\{0, 1, \ldots, n\}$, because a new leaf or root inherits all the labels on the subtree it came from. We replace these subsets by $0, 1, \ldots, r$, labelling the subsets *in increasing order of their minimal elements.*

The point of the labelling convention is that the conventional ordering introduced above is *compatible* with identifying a subtree to a single edge, provided one regards a subtree as enumerated when all its edges have been enumerated. For instance, a subtree containing the root is always labelled 0, and comes last in the conventional ordering.

Now we are ready to start defining our chain complex. To begin with we use reduced cubical chains, because $\tilde{T}_n$ is naturally a cubical complex.

Definition. Let K'_∞ be $\bigcup_{n=2}^{\infty} K'_n$, where the complexes K'_n are defined inductively as follows:

(1) K'_2 is the chain complex $C_*(\tilde{T}_2)$ of the one-point tree space $\tilde{T}_2$

(2) for $n \geq 3$, suppose that we have already defined the complex K'_{n-1} as a quotient of $\bigoplus_{2 \leq i < n} C_*(\tilde{T}_i)$. Then the complex K'_n is obtained by attaching $C_*(\tilde{T}_n)$ to K'_{n-1} along the subcomplex $C_*(T_n)$ of fully-grown trees. The attaching map $\varphi_{n-1} : C_*(T_n) \to K'_{n-1}$ takes the generator corresponding

to an n-tree t with fully-grown edge x to the class $(-1)^{n-r}t/x+(-1)^{r+1}t\backslash x$ in K'_{n-1}, where r is the number of leaves in t/x.

The previous identifications in K'_{n-1} ensure that the attaching map is well-defined and independent of the choice of the edge x, and evidently K_n is by construction a quotient of $\bigoplus_{2\leq i\leq n} C_*(\tilde{T}_i)$. The cubes t/x and $t\backslash x$ are of course labelled by the convention above, and oriented by the ordering convention. It should be noted that whenever t has more than one internal edge, at least one of the cubes t/x and $t\backslash x$ is a degenerate face.

Subdividing K'_∞. We shall show that there is a natural, geometrically-inspired contraction of the complex K'_∞. It is not easy to describe in terms of the cubical chains, because geometrically the image of $\tilde{T}_n$ is deformed through $\tilde{T}_{2n}$ in a way which is not cellular, but diagonal, on the cubes.

Therefore we replace each cubical complex $\tilde{T}_n$ by its natural simplicial subdivision, in which each r-cube is replaced by $r!$ r-simplices. (An n-tree belongs to one or other of these, depending upon which internal edges are longer than which others. Diagonal simplices in $\tilde{T}_n$ contain trees having certain edges of equal length.) Every cubical chain is a chain of the simplicial subdivision, so we have enlarged $C_*(\tilde{T}_n)$; and we make identifications among these exactly as before to obtain a chain complex K_∞, quasi-isomorphic to and containing K'_∞. But we continue to use cubes as blocks of simplices (sums of generators) in K_∞.

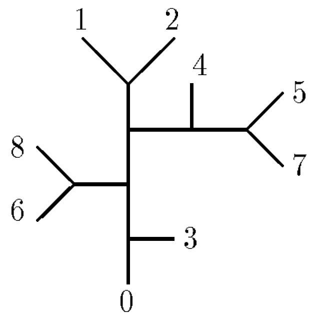
Informal description of the contraction. The contraction of K_∞ closely follows this geometrical idea. A labelled n-tree t passes through N stages $t_0, t_1, \ldots, t_N$ during the homotopy, where $t_0 = t$ and N is the total number of edges of t. In the tree t_i there are *two* identical copies of each of the first i edges in the conventional order, and one copy of the others. As identical edges must have the same length, t_i represents a diagonal cube in some $\tilde{T}_{n+j}$ having the same dimension as t. The homotopy connecting t_{i-1} and t_i is represented by a tree Δ_i like t_i but with one new edge below the two copies of the ith edge, connecting the most recently-doubled edge to the undoubled part. This is a cube of dimension one higher. Shrinking one undoubled edge, or two identical edges, to a point is a cubical face operator: therefore Δ_i has t_{i-1} and t_i as faces. Finally, t_N is the sum of two copies of t. A more formal description follows.

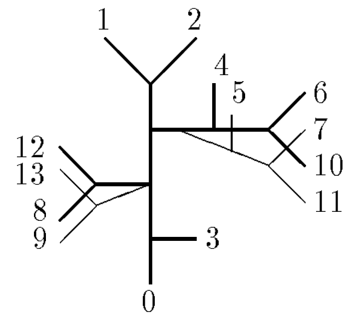
The double of a tree. Let t be an n-tree. The *double* $Y(t)$ is the $2n$-tree obtained by taking two identical copies t' and t'' of the tree t, and grafting them by the roots onto the two leaves of the unique tree in $\tilde{T}_2$. Pairs of identical edges have the same length. We label the result as follows. The two leaves formerly labelled i are marked $i-\frac{1}{2}$ and i in t' and t'' respectively. Then all labels are multiplied by two to give integers.

The construction Δ_i. We actually define $\Delta_i(t)$ and t_i by induction on i. We set $t_0 = t$. If t_{i-1} has been defined, and x_i is the ith edge of t in the conventional ordering, we define $\Delta_i(t)$ to be the result of grafting the double $Y(t/x_i)$ by its root onto the leaf x_i of $t_{i-1}\backslash x_i$. We define t_i by shrinking the grafted internal edge (formerly the root of the double $Y(t/x_i)$) of $\Delta_i(t)$ to a point. It follows from the inductive definition that t_i contains two copies of edges $x_1, \ldots, x_i$ and one copy of the higher-numbered edges. We note that $\Delta_i(t)$ and t_i have been defined cube by cube, or a block of generators of K_n at a time.

We have to label t_i and $\Delta_i(t)$. As for the doubling construction, we give the two copies of the leaf formerly designated i the labels $i - \frac{1}{2}$ and i, without changing the labels on the undoubled leaves. Then we replace the labels in bijective order-preserving fashion with the integers $1, 2, \ldots, s$ for some s.

Example. t $\Delta_8(t)$





Definition. If t is a cube corresponding to a tree-shape with a total of N edges, we define $\Delta(t) = \sum_{i=1}^{N} (-1)^i \Delta_i(t)$.

We claim that this defines a contracting homotopy of K_∞ by specifying it on the generating simplices, a cubical block at a time. To prove this, we must verify that $\partial\Delta + \Delta\partial = 1 - \pi$, where $\pi : K_\infty \to K_\infty$ factors through the chain complex of a point. So we have to investigate how Δ commutes with respect to face relations. This includes verifying that Δ respects the identifications used to define K_∞.

As we are still working with cubical blocks in K_∞, even though some of them may be diagonal cubes with certain coordinates equal, it is the cubical face operators we have to check. Let x_i be the ith edge of a tree t corresponding to a certain cube, also denoted t, in K_∞. If x_i is an internal edge, there is a face operator ∂_i corresponding to shrinking the length of x_i (and of all edges forced to have the same length) to zero. When the length of x_i stretches to 1, we have the opposite face ε_i of the cube, which by construction of K_∞ is identified with $t/x_i + t\backslash x_i$ (which is the zero chain when x_i lies between two internal edges of t).

By checking the geometrical details, we can now verify a whole slew of "cubical identities" such as (to give one instance)

$$\varepsilon_i \Delta_j(t) = \ \pm\, t/x_i \ \pm\, t/x_i + \Delta_{j-f+1}(t\backslash x_i)$$

when x_i is an edge of t/x_j (which implies $i \leq j$) and where f denotes the number of edges of t/x_i. The enumeration of faces is more complex here than in the case of the usual simplicial or cubical identities, because of the branching of trees. But some of the formulae simply assert that a certain face is degenerate, and is therefore a zero chain. For instance, the above formula gives a non-zero right hand side only in two cases: first, when x_j is the root of t and $i = j$; second, when x_i has nothing but leaves above it.

In calculating these identities it is essential to remember that ∂_i affects identical edges simultaneously and not separately, and likewise for ε_i.

The cubical identities in full. Let x_i and x_j be edges of t and let f be the number of edges of t/x_i. We denote the number of free edges (leaves plus root) of a tree s by $l(s)$. Then

(1) If $i \leq j$ and x_i is an internal edge or the root of t/x_j

$$\varepsilon_i \Delta_j(t) = (-1)^{l(\Delta_j(t)\backslash x_i)} t/x_i + (-1)^{l(\Delta_j(t)\backslash x_i)} t/x_i + \Delta_{j-f+1}(t\backslash x_i) \ .$$

(2) If $i < j$ and x_i is an internal edge of $t\backslash x_j$

$$\varepsilon_i \Delta_j(t) = (-1)^{l(\Delta_j(t)\backslash x_i)} t_{j-1}/x_i + \Delta_{j-f+1}(t\backslash x_i) \ .$$

(3) If $i \geq j$, x_i is an internal edge of t and x_j is the qth edge of t/x_i

$$\varepsilon_{i+1} \Delta_j(t) = (-1)^{l(\Delta_j(t)/x_i)} t_{j-1}\backslash x_i + (-1)^{l(\Delta_j(t)\backslash x_i)} \Delta_q(t/x_i) \ .$$

(4) For all i such that x_i is a leaf of t

$$\varepsilon_{i+1} \Delta_i(t) = -t_{i-1} + (-1)^{l(t_{i-1})} \Delta_1(t/x_i)$$

(5) If $i > j$, x_i is an internal edge of t and x_j is the pth edge of $t\backslash x_i$

$$\varepsilon_{i+1} \Delta_j(t) = (-1)^{l(\Delta_j(t)\backslash x_i)} t_{j-1}/x_i + (-1)^{l(\Delta_j(t)/x_i)} \Delta_p(t\backslash x_i) \ .$$

(6) If $i < j$ and x_i is an internal edge of t

$$\partial_i \Delta_j(t) = \Delta_{j-1}(\partial_i t) \ .$$

(7) For all i

$$\partial_i \Delta_i(t) = t_{i-1} = \partial_i \Delta_{i-1}(t) \ .$$

(8) If $i > j$ and x_i is an internal edge of t

$$\partial_{i+1} \Delta_j(t) = \Delta_j(\partial_i t) \ .$$

(9) If $i \leq j$

$$\Delta_i \Delta_j = \Delta_{j+1} \Delta_i \ .$$

The first five identities, together with the labelling convention, imply that Δ is compatible with the identifications used in defining K_∞, and is therefore well-defined. The fourth identity gives, according to the dimension of the cube t,

$$\varepsilon_2 \Delta_1(t) = \begin{cases} -t & \text{if } \dim t > 0 \\ -t + (-1)^{l(t)} *_2 & \text{if } \dim t = 0, \end{cases}$$

where $*_2$ is the unique 2-tree. The first identity gives, when x_N is the last edge (root) of t

$$\varepsilon_N \Delta_N(t) = \begin{cases} (-1)^{l(t)+1} 2t & \text{if } \dim t > 0 \\ (-1)^{l(t)+1} 2t + *_2 & \text{if } \dim t = 0. \end{cases}$$

¿From the cubical identities, it follows that Δ is a chain homotopy from $1 + \pi$, where 1 is the identity map and π is a point map as above, to twice the identity map. (There is additional checking to be done on 0-chains; π is given by $\pi(*_n) = (-1)^n (n-1) *_2$, where $*_n$ denotes the star tree with n leaves.) Therefore $1 - \pi$ is nullhomotopic by the chain homotopy Δ, and K_∞ is contractible.

Acyclicity of $\mathcal{K}_A(A;M)$

The chain complex $\mathcal{K}_A(A;M)$ is constructed as the cofibre of a map from a partial realization $|\mathcal{M}|'$ to M. It very easily follows that $\mathcal{K}_A(A;M)$ is acyclic if $|\mathcal{M}|'$ is contractible; so this contractibility is what we have to prove. It is sufficient to treat the case when the coefficient module M is A.

We describe $|\mathcal{M}|'$. Just as K_∞ in the previous section was constructed by glueing together the tree spaces $\tilde{T}_n$ according to certain labelling conventions, so $|\mathcal{M}|'$ is obtained by glueing together the spaces $\mathcal{C}_n/\Sigma_n$ of a cofibrant cyclic E_∞ operad $\mathcal{C}$, for which we shall use the tree operad. (In the new context, it can be seen that the somewhat arbitrary labelling convention is actually quite immaterial: a different choice leads by conjugation in symmetric groups to homotopic glueing maps, and so to a quasi-isomorphic result. But a choice has to be made.)

Thus $|\mathcal{M}|'$ is an extended version of K_∞, incorporating the actions of the symmetric groups. One tries to contract it by applying fibrewise the contraction of K_∞. This amounts to constructing a coherent system of higher homotopies among the contractions obtained by twisting the original contraction by all elements of the symmetric group. One expects to be able to do this since, if Δ and Ξ are two contractions of a complex, then $\Xi\Delta$ is a homotopy of homotopies from Δ to Ξ.

Construction of $|\mathcal{M}|'$. We construct $|\mathcal{M}|'$ using the cofibrant tree operad $\mathcal{T}$ of 1.5. Since the symmetric group Σ_n acts trivially on the nth tensor power of A over itself, the realization is built by glueing together the complexes $\mathcal{T}_n/\Sigma_n$. The free chain complex $\mathcal{T}_n/\Sigma_n$ has generators

$$[\sigma_1|\sigma_2|\dots|\sigma_k]\otimes t$$

in dimension $k+\dim t$, where $k\geq 0$, $\sigma_1,\dots,\sigma_k\in\Sigma_n$, and t is a simplex (or cube) of the tree space $\tilde{T}_n$. The boundary is given by

$$\begin{aligned}\partial([\sigma_1|\sigma_2|\dots|\sigma_k]\otimes t) \quad = \quad & [\sigma_2|\dots|\sigma_k]\otimes t \\ + \quad & \sum_{j=1}^{k-1}(-1)^j[\sigma_1|\dots|\sigma_j\sigma_{j+1}|\dots|\sigma_k]\otimes t \\ + \quad & (-1)^k[\sigma_1|\dots|\sigma_{k-1}]\otimes\sigma_k t \\ + \quad & (-1)^{k+1}[\sigma_1|\sigma_2|\dots|\sigma_k]\otimes\partial t\end{aligned}$$

where ∂t is the boundary in $\tilde{T}_n$, and $\sigma_k t$ is defined using the permutation action of Σ_n on the labels of $\tilde{T}_n$.

The identifications which form $|\mathcal{M}|'$ from the chain complexes $\mathcal{T}_n/\Sigma_n$ mirror those used to form K_∞ from the $\tilde{T}_n$. In the latter case, we recall, when an internal edge x_j of t has length 1, the chain t of K_∞ is identified with $t/x_j+t\backslash x_j$, which are trees labelled by our convention. This labelling convention is sufficiently functorial to allow us to identify, when x_j has length 1, the chain $[\sigma_1|\dots|\sigma_k]\otimes t$ with

$$[\hat{\sigma}_1|\dots|\hat{\sigma}_k]\otimes t/x_j \quad + \quad [\check{\sigma}_1|\dots|\check{\sigma}_k]\otimes t\backslash x_j$$

where $\hat{\sigma}_i$ and $\check{\sigma}_i$ are the induced permutations of conventional labelling sets for t/x_j and $t\backslash x_j$. For instance, if $\hat{\sigma}_k,\dots,\hat{\sigma}_{i+1}$ have already been defined, then $\hat{\sigma}_i$ is uniquely determined by the stipulation that $\hat{\sigma}_i\hat{\sigma}_{i+1}\dots\hat{\sigma}_k(t/x_j)$ be the conventional labelling of $(\sigma_i\sigma_{i+1}\dots\sigma_k t)\,/\,x_j$. By these means we can define cubical face operators ε_j in $|\mathcal{M}|'$ just as in K_∞.

In a totally analogous way we can extend the definition of the operators Δ_j to $|\mathcal{M}|'$, setting

$$\Delta_j\left([\sigma_1|\dots|\sigma_k]\otimes t\right) \quad = \quad [\bar{\sigma}_1|\dots|\bar{\sigma}_k]\otimes\Delta_j(t)$$

where $[\bar{\sigma}_1|\dots|\bar{\sigma}_k]$ is the induced string of permutations of the conventional labelling set of $\Delta_j(t)$. We define Δ to be the alternating sum $\sum(-1)^j\Delta_j$, but we can not expect this to be a contraction, because of the form of the boundary operator in $|\mathcal{M}|'$. Nor is it true that $\bar{\sigma}_i\Delta = \Delta\sigma_i$, because the action of Σ_n does not preserve the conventional ordering which is essentially used in the definition of Δ.

In the following definition and all that follows, we use the notation $\bar{\sigma}$ to denote any permutation induced by σ on a set of tree labels derived by our conventions. The context always implies exactly what the trees in question are, so it is not necessary to burden the notation with any heavy details.

Definition. We define an operator $\tilde{\Delta}$ on the chains of $|\mathcal{M}|'$ by setting

$$\tilde{\Delta}([\sigma_1|\dots|\sigma_k]\otimes t) \quad = \quad \sum_{j=0}^{k}\left\{(-1)^j[\bar{\sigma}_1|\dots|\bar{\sigma}_j]\otimes\Delta\bar{\sigma}_{j+1}\Delta\bar{\sigma}_{j+2}\Delta\dots\Delta\bar{\sigma}_k\Delta(t)\right\}\ .$$

Theorem. *The chain complex $\mathcal{K}_A(A;M)$ is acyclic.*

Proof. We have remarked above that it is sufficient to prove that $|\mathcal{M}|'$ is contractible, and that we may take M to be A. We simply claim that the homotopy $\tilde{\Delta}$, defined above, is a contraction of $|\mathcal{M}|'$.

To see this, one repeatedly uses the relation $\partial\Delta+\Delta\partial = 1-\pi$ in K_∞ to calculate that when t is a tree with at least one internal edge, the relation

$$\begin{aligned}
\partial(\Delta\bar{\sigma}_{j+1}\Delta\bar{\sigma}_{j+2}\Delta\dots\Delta\bar{\sigma}_k\Delta(t)) \quad &= \quad \bar{\sigma}_{j+1}\Delta\bar{\sigma}_{j+2}\Delta\dots\Delta\bar{\sigma}_k\Delta(t)\\
&+ \quad \sum_{r=1}^{k-j-1}(-1)^r\Delta\bar{\sigma}_{j+1}\Delta\dots\Delta\bar{\sigma}_{j+r}\bar{\sigma}_{j+r+1}\Delta\dots\Delta\bar{\sigma}_k\Delta(t)\\
&+ \quad (-1)^{k-j}\Delta\bar{\sigma}_{j+1}\Delta\dots\Delta\sigma_k t\\
&+ \quad (-1)^{k-j-1}\Delta\bar{\sigma}_{j+1}\Delta\dots\Delta\bar{\sigma}_k\Delta(\partial t)
\end{aligned}$$

holds in $|\mathcal{M}|'$, and a minor variant when t is a star-tree. Then straightforward calculation with the formulae defining ∂ and $\tilde{\Delta}$ gives

$$\partial\tilde{\Delta}+\tilde{\Delta}\partial = \begin{cases} 1 & \text{in dimension} > 0,\\ 1-\pi & \text{in dimension } 0,\end{cases}$$

where π is a point map. The theorem is therefore proved. □

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