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Gamma homology, Lie representations and E_∞ multiplications

Alan Robinson

Mathematics Institute, University of Warwick, Coventry CV4 7AL, England
(e-mail: car@maths.warwick.ac.uk)

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Abstract. We prove that the stable homotopy of any Γ-module F is the homology of a bicomplex $\Xi(F)$, in which the $(q-1)$st row is the two-sided bar construction $\mathcal{B}(\mathrm{Lie}_q^*, \Sigma_q, F[q])$. This gives a natural homotopical cotangent bicomplex for graded commutative algebras, in a form suitable for use in a new obstruction theory for classifying E_∞ ring structures on spectra. The E_∞ structure on certain Lubin-Tate spectra is a corollary.

Introduction

Let F be any functor from finite based sets to spaces. The stable homotopy groups of F, which we denote by $\pi_*(F)$, are defined as the homotopy groups of the spectrum $\|F\|$ associated with F by the well known construction introduced by G. Segal.

We consider the discrete abelian case, where F takes values in k-modules for some commutative ring k. In Sect. 3 we give an explicit functorial bicomplex $\Xi(F)$ the homology of which is $\pi_*(F)$. The proof is an application of recent theory due to Pirashvili and Richter [10], [11].

In Sect. 4 we apply the Ξ-construction to the functor which takes a based set S_+ to the module $B^{\otimes S} \otimes M$, where B is a given graded commutative algebra and M a B-module. This gives a canonical, functorial cotangent complex for graded commutative algebras. Its homology is the Γ-homology $H\Gamma_*(B|A; M)$, where A is the ground ring. Theorem 4.2 reconciles this construction with other definitions of Γ-homology, E_∞-homology and topological André-Quillen homology for commutative rings as defined by Basterra, Kriz and others (see [2], [17]). The description of Γ-homology as stable homotopy yields some calculations [14], [15].

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Our interest in the Ξ-construction was inspired by a question in homotopy theory: when does a ring spectrum admit a multiplication satisfying the E_∞ homotopy associativity and commutativity conditions? In the language of S-modules, what conditions are sufficient for enhancing a multiplication in the homotopy category of S-modules to a commutative S-algebra structure? In Sect. 5 we answer this question by setting up an obstruction theory based upon Γ-homology. The Ξ-complex arises there in a natural way. Other very different approaches to this problem have been given by Basterra [1] and Kriz, and in unpublished work by Goerss and Hopkins.

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1. Γ-modules

Let Γ denote the category of finite based sets and basepoint-preserving maps. A construction of G. Segal [18], refined by Bousfield and Friedlander [3], assigns a spectrum $\|F\|$ to each functor F from Γ to based simplicial sets. By applying F to the set of simplices at each level in a simplicial model of S^n, Bousfield and Friedlander obtain a bisimplicial set. The diagonal simplicial set of this is the nth space in the spectrum $\|F\|$. The homotopy groups of the spectrum $\|F\|$ are called the (stable) homotopy groups of F. We denote them by $\pi_* F$.

We are concerned with the discrete abelian case. Let us fix a commutative ring k with unit, to be our ground ring. A *(left)* Γ*-module* is a functor $\Gamma \to$ *k-mod*. Since a k-module can be regarded as a discrete simplicial set, this fits into the above context. If F is a Γ-module, then the spaces in the spectrum $\|F\|$ are simplicial k-modules. The theory of Γ-modules and their homotopy has been substantially developed in recent work of Teimuraz Pirashvili and Birgit Richter [10], [13].

The most fundamental Γ-module is the functor L which assigns to each based set X the free k-module $kX/k\{0\}$ generated by X, modulo the submodule generated by the basepoint. On simplicial sets, L is the reduced chain functor. Let us consider the case of a Γ-module which has the particular form $F = \Psi \circ L$, where Ψ : *k-mod* $\to$ *k-mod* is any functor. From the fact that $L(S^n)$ is a projective resolution of (k, n) in the sense of [4] it follows that the stable homotopy groups $\pi_*(\Psi \circ L)$ are the stable derived functors of Ψ in the sense of Dold and Puppe, evaluated on k.

The main result of [11] gives an explicit chain complex for calculating the stable homotopy of any Γ-module. This complex is based upon the nerve of the subcategory of all surjections in Γ. Consequently it is quite large, even when Γ is replaced (as it usually is) by a minimal skeleton with one set of each cardinality. In Sect. 2 we give a much smaller complex $\Xi(F)$, based upon the Lie operad, which has the same property.

2. The modules Lie_n and the standard bicomplex

As we turn to the combinatorial details, we replace the category Γ by the equivalent full subcategory in which the objects are the finite based sets $[n] = \{0, 1, 2, \ldots, n\}$ for all $n \geq 0$, where $0 \in [n]$ is the basepoint. We shall denote this subcategory henceforth by Γ.

The group of automorphisms in Γ of the object $[n]$ is the permutation group Σ_n. The important subcategory of all surjections in Γ is generated by these automorphisms together with the special surjections $c_{ij} : [n] \to [n-1]$ which are defined for $0 \leq i < j \leq n$ by the formulae

$$c_{ij}(t) = \begin{cases} t & \text{for } 0 \leq t < j \\ i & \text{for } t = j \\ t-1 & \text{for } j < t \leq n. \end{cases} \tag{2.0}$$

Thus c_{ij} maps j to i and is strictly order-preserving on $[n] \setminus \{j\}$.

2.1. The Lie representations

Let $\mathcal{L}_n$ be the free Lie algebra over k on the set of generators $\{x_i\}_{1\leq i\leq n}$. We denote by Lie_n the so-called *multilinear part* of $\mathcal{L}_n$. This can be described in many different ways. First, it is defined as the direct summand of $\mathcal{L}_n$ spanned by all Lie monomials containing each of the n generators exactly once. Second, it is the nth module in the Lie operad. Third, it is isomorphic to the module of all natural transformations $\Phi^{\otimes n} \to \Phi$, where Φ is the forgetful functor from Lie algebras (over k) to k-modules.

The symmetric group Σ_n acts upon Lie_n by permuting the n generators. The $k\Sigma_n$-module thus obtained is known as the *Lie representation.* It has many applications in combinatorics, geometry and homotopy theory. Very often it occurs twisted by the sign character, and this is the version we require. We define the left action of Σ_n on the k-module Lie_n by setting

$$\sigma \cdot f(x_1, \ldots, x_n) = \varepsilon(\sigma)\, f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for every multilinear Lie monomial f and every $\sigma \in \Sigma_n$, where $\varepsilon(\sigma)$ is the sign of σ. Let Lie_n^* be the dual k-module $\mathrm{Hom}(\mathrm{Lie}_n, k)$, which is thus a right Σ_n-module.

We shall require the following properties of Lie_n (see [20, 2.3]):

(1) the left regulated Lie brackets

$$\sigma \cdot [x_1, [x_2, [x_3, \ldots, [x_{n-1}, x_n]..]]] \quad \text{for} \quad \sigma \in \Sigma_{n-1}$$

form a k-basis of Lie_n. Therefore

(2) the k-modules Lie_n and Lie_n^* are free of rank $(n-1)!$, and

(3) the restricted Σ_{n-1}-modules $\mathrm{Res}^{\Sigma_n}_{\Sigma_{n-1}}\mathrm{Lie}_n$ and $\mathrm{Res}^{\Sigma_n}_{\Sigma_{n-1}}\mathrm{Lie}_n^*$ are respectively isomorphic to the left and right regular representations.

2.2. The homomorphisms $\gamma_{ij} : \mathrm{Lie}_n^* \to \mathrm{Lie}_{n-1}^*$

We now introduce certain linear maps $\gamma_{ij} : \mathrm{Lie}_n^* \to \mathrm{Lie}_{n-1}^*$. These are induced by the standard surjections in Γ, but they do not make the duals of the Lie representations into a Γ-module, as 2.4 below makes clear.

Since Lie_n^* is for each n a finitely-generated projective k-module, we can define the γ_{ij} completely by giving their duals $\gamma_{ij}^* : \mathrm{Lie}_{n-1} \to \mathrm{Lie}_n$. For $0 \le i < j \le n$ we set

$$\big(\gamma_{ij}^* f\big)(x_1, \dots, x_n) = \begin{cases} (-1)^{j+1}[x_j, f(x_1, .., \hat{x}_j, .., x_n)] & \text{if } i = 0 \\ (-1)^{j+1} f(x_1, .., x_{i-1}, [x_i, x_j], x_{i+1}, .., \hat{x}_j, .., x_n) & \text{if } i > 0 \end{cases}$$

for each multilinear Lie monomial $f(x_1, \dots, x_{n-1})$, where the circumflex accent ˆ means that the accented term is omitted.

2.3. The standard relations

It follows from the multilinearity of f and from the Jacobi identity that for $i < j < k$

$$\begin{aligned} \big(\big(\gamma_{0j}^*\gamma_{0,k-1}^* + \gamma_{0k}^*\gamma_{0j}^*\big) f\big)(x_1, \dots, x_n) & \\ &= (-1)^{j+k+1}[x_j, [x_k, f(x_1, .., \hat{x}_j, .., \hat{x}_k, .., x_n)]] \\ &\quad + (-1)^{j+k}[x_k, [x_j, f(x_1, .., \hat{x}_j, .., \hat{x}_k, .., x_n)]] \\ &= (-1)^{j+k+1}[[x_j, x_k], f(x_1, .., \hat{x}_j, .., \hat{x}_k, .., x_n)] \\ &= -\ \big(\gamma_{jk}^*\gamma_{0j}^* f\big)(x_1, \dots, x_n) \end{aligned}$$

and that

$$\begin{aligned} \big(\gamma_{ij}^*\gamma_{i,k-1}^* + \gamma_{ik}^*\gamma_{ij}^* + \gamma_{jk}^*\gamma_{ij}^*\big) f(x_1, \dots, x_n) & \\ &= (-1)^{j+k+1} f(x_1, \dots, [[x_i, x_j], x_k] - [[x_i, x_k], x_j] \\ &\quad - [x_i, [x_j, x_k]], .., \hat{x}_j, .., \hat{x}_k, .., x_n) \\ &= 0\ . \end{aligned}$$

Passing to the dual in the above gives all cases ($i = 0$ and $i > 0$) of the first relation in the following lemma. The other relations are even more straightforward to verify.

2.4 Lemma. *When* $i < j < k < l$

$$\begin{aligned} \gamma_{ij}\gamma_{jk} + \gamma_{ij}\gamma_{ik} + \gamma_{i,k-1}\gamma_{ij} &= 0 \\ \gamma_{ij}\gamma_{kl} + \gamma_{k-1,l-1}\gamma_{ij} &= 0 \\ \gamma_{ik}\gamma_{jl} + \gamma_{j,l-1}\gamma_{ik} &= 0 \\ \gamma_{i,l-1}\gamma_{jk} + \gamma_{jk}\gamma_{il} &= 0\ . \end{aligned}$$

□

2.5. The bicomplex $\Xi(F)$

Let F be any Γ-module. Then $F[n]$ is a left Σ_n-module by restriction of structure, which we can combine with the right Σ_n-module Lie_n^* in the two-sided bar construction $\mathcal{B}(\mathrm{Lie}_n^*,\ \Sigma_n,\ F[n])$. We take this as the $(n-1)$st row in our standard bicomplex, so that we have

$$\Xi_{p,q}(F) = \mathrm{Lie}_{q+1}^* \otimes k\big[\Sigma_{q+1}^p\big] \otimes F[q+1]\ .$$

the horizontal differential ∂' being the alternating sum $\sum_{i=0}^{p}(-1)^i\partial_i'$ of face operators:

$$\begin{aligned}\partial'(z\otimes[\sigma_1|\sigma_2|\dots|\sigma_p]\otimes y) = {} & z\sigma_1\otimes[\sigma_2|\dots|\sigma_p]\otimes y + \\ & + \sum_{i=1}^{p-1}(-1)^i(z\otimes[\sigma_1|\sigma_2|\dots|\sigma_i\sigma_{i+1}|\dots|\sigma_p]\otimes y) + \\ & + (-1)^p(z\otimes[\sigma_1|\sigma_2|\dots|\sigma_{p-1}]\otimes\sigma_p y)\ .\end{aligned}$$

We define a vertical differential $\partial'' : \Xi_{p,q} \longrightarrow \Xi_{p,q-1}$, that is

$$\mathrm{Lie}_{q+1}^*\otimes k\big[\Sigma_{q+1}^p\big]\otimes F[q+1] \longrightarrow \mathrm{Lie}_q^*\otimes k\big[\Sigma_q^p\big]\otimes F[q]$$

in terms of the operators c_{ij} (see (2.0)) in the Γ-structure on F, and the homomorphisms γ_{ij} of 2.2, by setting

$$\begin{aligned}(-1)^p\,\partial''(z\otimes&[\sigma_1|\sigma_2|\dots|\sigma_p]\otimes y)\\ &= \sum_{0\le s<t\le q+1}\gamma_{st}z\otimes c_{st}[\sigma_1|\sigma_2|\dots|\sigma_p]\otimes c_{(\sigma_1\dots\sigma_p)^{-1}\{s,t\}}\,y\end{aligned}$$

where $c_{st}[\sigma_1|\sigma_2|\dots|\sigma_p]$ is the p-simplex of the nerve of Σ_q which forms the bottom row of the commutative diagram

$$\begin{array}{ccccccccc} [q+1] & \xrightarrow{\sigma_p} & [q+1] & \xrightarrow{\sigma_{p-1}} \cdots \xrightarrow{\sigma_2} & [q+1] & \xrightarrow{\sigma_1} & [q+1] \\ c_{(\sigma_1\dots\sigma_p)^{-1}\{st\}}\downarrow & & \downarrow c_{(\sigma_1\dots\sigma_{p-1})^{-1}\{st\}} & & \downarrow c_{\sigma_1^{-1}\{st\}} & & \downarrow c_{st} \\ [q] & \longrightarrow & [q] & \longrightarrow\cdots\longrightarrow & [q] & \longrightarrow & [q] \end{array}$$

arising as follows. Given an isomorphism σ_1 in the category Γ and a surjection c_{st} as here, the right-hand square above can be uniquely completed with a surjection $c_{\sigma_1^{-1}\{st\}}$ and an isomorphism $c_{st}[\sigma_1] : [q] \to [q]$. Repeating this procedure p times constructs the whole diagram, and the simplex $c_{st}[\sigma_1|\sigma_2|\dots|\sigma_p]$. (The notation c_{st} when $s > t$ means c_{ts}, and $c_{\sigma\{st\}}$ is an abbreviation for $c_{\sigma(s),\sigma(t)}$.)

2.6 Proposition. *$\Xi(F)$ is a bicomplex.*

Proof. The relation $\partial'\partial' = 0$ holds because the two-sided bar resolution (see for instance [19, §2.3]), is a chain complex. On the other hand $\partial''\partial'' = 0$ because ∂'' is on the first tensor factor the Leibniz differential ([8, 10.6.2]) on a generic Lie algebra. The vanishing of $\partial''\partial''$ can also be explicitly verified by a calculation using Lemma 2.4.

Finally, we must verify that $\partial'\partial'' + \partial''\partial' = 0$. In fact, ∂'' anticommutes with each of the horizontal face operators ∂'_i. For all faces except ∂'_0 this is an immediate consequence of the commutative diagram used above to define $c_{st}[\sigma_1|\sigma_2|\ldots|\sigma_p]$. The face ∂'_0 however involves the action of σ_1 on Lie_{q+1}. We first note that the signs of σ_1 and of $c_{\sigma_1\{st\}}(\sigma_1)$ are related by $\varepsilon(c_{\sigma_1\{st\}}(\sigma_1)) = (-1)^{t-\sigma_1(t)}\varepsilon(\sigma_1)$. For any $f \in \mathrm{Lie}_q$ we therefore have

$$\begin{aligned}
&\sigma_1(\gamma^*_{0t}f)(x_1, \ldots, x_{q+1}) \\
&\qquad = (-1)^t\varepsilon(\sigma_1)[x_{\sigma_1(t)}, f(x_{\sigma_1(1)}, .., \hat{x}_{\sigma_1(t)}, .., x_{\sigma_1(q+1)})] \\
&\qquad = (-1)^t\varepsilon(\sigma_1)\varepsilon(c_{\sigma_1\{0t\}}\sigma_1)[x_{\sigma_1(t)}, (c_{\sigma_1\{0t\}}\sigma_1)\cdot f(x_1, .., \hat{x}_t, .., x_{q+1})] \\
&\qquad = \gamma^*_{0,\sigma_1(t)}(c_{\sigma_1\{0t\}}\sigma_1)\cdot f(x_1, .., x_{q+1})
\end{aligned}$$

which proves that $\sigma_1\gamma^*_{0t} = \gamma^*_{0,\sigma_1(t)}(c_{\sigma_1\{0t\}}\sigma_1)$. Since γ_{0t} is defined as the adjoint of γ^*_{0t}, we have $\gamma_{0t}(z\sigma_1) = \gamma_{0,\sigma_1(t)}(z)\, c_{\sigma_1\{0t\}}(\sigma_1)$ for all $z \in \mathrm{Lie}^*_q$. We show in the same way that $\gamma_{st}(z\sigma_1) = \gamma_{\sigma_1(s,t)}(z)\, c_{\sigma_1\{st\}}(\sigma_1)$ when $s > 0$, though here we have to separate the cases $\sigma_1(s) < \sigma_1(t)$ and $\sigma_1(s) > \sigma_1(t)$. The relations just established are precisely those needed to prove that ∂'' anticommutes with the remaining face operator ∂'_0. Thus $\partial'\partial'' + \partial''\partial' = 0$, and the lemma is proved. □

3. The complex $\mathrm{Tot}\,\Xi$ and stable homotopy

In this section we use the theory of [10] and [11] to prove that the homology of the Ξ-complex is stable homotopy.

3.1 Definition. *We denote by $H\Xi_*(F)$ the homology of the total complex* $\mathrm{Tot}\,\Xi(F)$.

Let L be the Γ-module (denoted t^* in [10]) which assigns to $[n]$ the free k-module generated by $[n]$ modulo the submodule generated by the basepoint:

$$L[n] = k[n]/k[0]\ .$$

We recall that tensor products over k of Γ-modules are defined objectwise:

$$(F \otimes G)[n] = F[n] \otimes G[n]\ .$$

3.2 Lemma. *For any Γ-module F*

$$H\Xi_i(F \otimes L) \approx \begin{cases} F[0] & \text{for } i = 0 \\ 0 & \text{for } i \neq 0\,. \end{cases}$$

Proof. We calculate the homology of the bicomplex $\Xi(F \otimes L)$ using the spectral sequence which first calculates the homology of the rows. The $(n-1)$st row is the bar construction $\mathcal{B}(\mathrm{Lie}_n^*, \Sigma_n, F[n] \otimes L[n])$, so the horizontal homology $E^1_{n-1,*}$ is $\mathrm{Tor}_*^{\Sigma_n}(\mathrm{Lie}_n^*, F[n] \otimes L[n])$. Let us denote the trivial representation of any group by η. From the observation that $L[n]$ is the induced Σ_n-module $\mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n} \eta$, we have

$$F[n] \otimes L[n] \approx F[n] \otimes \mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n} \eta \approx \mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n}\Big(\mathrm{Res}_{\Sigma_{n-1}}^{\Sigma_n} F[n] \otimes \eta\Big)$$
$$\approx \mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n} \mathrm{Res}_{\Sigma_{n-1}}^{\Sigma_n} F[n] \ .$$

Using this and the result §2.1(3) that Lie_n^* restricts to the regular representation $\mathrm{Ind}_{\Sigma_1}^{\Sigma_{n-1}} \eta$ of Σ_{n-1}, we can reduce the groups $E^1_{n-1,*}$ in two stages: first to the homology of Σ_{n-1}, and then to that of the trivial group Σ_1:

$$\begin{aligned} E^1_{n-1,*} &\approx \mathrm{Tor}_*^{\Sigma_n}(\mathrm{Lie}_n^*,\ F[n] \otimes L[n]) \\ &\approx \mathrm{Tor}_*^{\Sigma_n}\left(\mathrm{Lie}_n^*,\ \mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n} \mathrm{Res}_{\Sigma_{n-1}}^{\Sigma_n} F[n]\right) \\ &\approx \mathrm{Tor}_*^{\Sigma_{n-1}}\left(\mathrm{Res}_{\Sigma_{n-1}}^{\Sigma_n} \mathrm{Lie}_n^*,\ \mathrm{Res}_{\Sigma_{n-1}}^{\Sigma_n} F[n]\right) \\ &\approx \mathrm{Tor}_*^{\Sigma_{n-1}}\left(\mathrm{Ind}_{\Sigma_1}^{\Sigma_{n-1}} \eta,\ \mathrm{Res}_{\Sigma_{n-1}}^{\Sigma_n} F[n]\right) \\ &\approx \mathrm{Tor}_*^k(k, F[n]) \ , \end{aligned}$$

which is $F[n]$ in degree zero, and zero elsewhere. Therefore the E^1-term reduces to the vertical edge, with $E^1_{0,n-1} \approx F[n]$.

We must calculate $d^1_{n,0} : F[n+1] \to F[n]$. Rewritten as

$$d^1_{n,0} : \mathrm{Lie}_{n+1}^* \otimes_{\Sigma_{n+1}} (F[n+1] \otimes L[n+1]) \to \mathrm{Lie}_n^* \otimes_{\Sigma_n} (F[n] \otimes L[n])$$

this is induced by the differential $\partial'' = \sum_{0 \le s < t \le n+1} (\gamma_{st} \otimes c_{st})$. We calculate the operators $\gamma_{st} \otimes c_{st}$ on $\mathrm{Lie}_{n+1}^* \otimes L[n+1]$ from their adjoints

$$c_{st}^* \otimes \gamma_{st}^* : L[n]^* \otimes_{\Sigma_n} \mathrm{Lie}_n \longrightarrow L[n+1]^* \otimes_{\Sigma_{n+1}} \mathrm{Lie}_{n+1} \ .$$

Let U_n denote the left regulated Lie bracket $[x_1, [x_2, [x_3, \dots, [x_{n-1}, x_n]..]]]$, which we shall denote more briefly by $[1, [2, [3, \dots, [n-1, n]..]]]$. The elements σU_n, for $\sigma \in \Sigma_{n-1}$, form a k-basis for Lie_n. The cosets $e_j = (j\ n)\Sigma_{n-1}$, for $1 \le j \le n$, form a k-basis of $L[n]^* \approx \Sigma_n/\Sigma_{n-1}$, and the single element $e_n \otimes U_n$ is a basis for the rank one k-module $L[n]^* \otimes_{\Sigma_n} \mathrm{Lie}_n$.

We take first the case $s = 0$. The definitions of c_{0t} and of γ_{0t}^* in Sects. 1–2 show that for $t < n+1$, the homomorphism $c_{0t}^* \otimes \gamma_{0t}^*$ maps $e_n \otimes U_n$ into

$$(-1)^{t+1} e_{n+1} \otimes [t, [1, [2, \dots, [t-1, [t+1, \dots, [n, n+1]..]]]]]$$

which equals $e_{n+1} \otimes U_{n+1}$ in $L[n+1]^* \otimes_{\Sigma_{n+1}} \mathrm{Lie}_{n+1}$ because the permutation $(t\ 1\ 2\ \dots t-1)$ needed to standardize the order of the basis elements in the

Lie bracket has sign $(-1)^{t+1}$. For $t = n+1$, from $c_{0,n+1}(n) = n$ we obtain

$$\begin{aligned}\big(c^*_{0,n+1}\otimes\gamma^*_{0,n+1}\big)(e_n \otimes U_n)&\\ &= (-1)^n e_n \otimes [n+1,[1,[2,\ldots,[n-1,n]..]]]\\ &= (-1)^n e_{n+1}\otimes (n\ n+1)[n+1,[1,[2,\ldots,[n-1,n]..]]]\\ &= (-1)^{n+1} e_{n+1}\otimes [n,[1,[2,\ldots,[n-1,n+1]..]]]\\ &= e_{n+1}\otimes U_{n+1}\end{aligned}$$

since $(n\ n+1)$ has sign -1 and the n-cycle $(1\ 2\ 3\ \ldots\ n)$ has sign $(-1)^{n+1}$. Therefore in all cases $(c^*_{0t}\otimes\gamma^*_{0t})(e_n\otimes U_n) = e_{n+1}\otimes U_{n+1}$. Similar calculations show that $(c^*_{st}\otimes\gamma^*_{st})(e_n\otimes U_n) = (-)^s 2e_{n+1}\otimes U_{n+1}$ for $s > 0$. These calculations of adjoint homomorphisms establish that the differential $d^1_{n,0} : F[n+1] \to F[n]$ has the following expression in terms of the Γ-operators on F:

$$d^1_{n,0} = \sum_{t=1}^{n+1} c_{0t} + 2\sum_{1\le s<t\le n+1} (-1)^s c_{st}\ .$$

We define an augmentation $E^1_{0,0} \approx F[1] \to F[0]$ by the same formula, which since $n = 1$ reduces at this level to the single term c_{01}.

Let $\kappa_n : [n] \to [n+1]$ be the morphism of Γ defined by $\kappa_n(i) = i$, for $1 \le i \le n$. The κ_n induce a sequence of morphisms in the Γ-module F which we also denote by κ. In Γ there are identities as follows: $c_{st}\kappa_n = \kappa_n c_{st}$ for $0 \le s < t \le n$, but $c_{s,n+1}\kappa_n = 1$ for $0 \le s \le n$. From this and the above formula for $d^1_{n,0}$ it follows that $\kappa_{n-1}d^1_{n-1,0} + d^1_{n,0}\kappa_n = (-1)^{n-1}$. This means that the morphisms $(-1)^n\kappa_n$ give a chain contraction of the augmented vertical edge $E^1_{*,0} \to F[0]$. Therefore $E^2_{p,q}$ is $F[0]$ when $(p,q) = (0,0)$, and is zero when $(p,q) \ne (0,0)$. The lemma is proved. □

3.3 Lemma. *If F is a constant Γ-module, then $\Xi(F)$ is acyclic. In fact, all the columns of $\Xi(F)$ are acyclic.*

Proof. It is clearly enough to show that all the columns are acyclic when F is the Γ-module with constant value k. Since all the modules $\Xi_{pq}(k) = \mathrm{Lie}^*_{q+1}\otimes k[\Sigma^p_{q+1}]$ are finite and free, it suffices to show that the complexes dual to the columns are acyclic.

The dual of the column $\Xi_{0,*}(k)$ at the edge has modules $\Xi^*_{0,q}(k) = \mathrm{Lie}_{q+1}$ and differential $\partial''^* = \sum_{0\le s<t\le q+1}\gamma^*_{st}$. By 2.1(3) the set of left regulated Lie brackets $[\alpha_1,[\alpha_2,..,[\alpha_q,q+1]..]]$ where $\alpha\in\Sigma_q$ forms a k-basis of $\Xi^*_{0,q}$. We define linear maps $\Delta^* : \Xi^*_{0,q} \to \Xi^*_{0,q-1}$ by the formula

$$\Delta^*[\alpha_1,[\alpha_2,..,[\alpha_q,q+1]..]] \quad = \quad \begin{cases} 0 & \text{if } \alpha_1 \ne 1\\ (\beta_2,[\beta_3,..,[\beta_q,\ q]..]) & \text{if } \alpha_1 = 1,\end{cases}$$

where $\beta_j = \alpha_j - 1$. From this definition it follows that $\Delta[1, g(2, \ldots, q+1)] = g(1, \ldots, q)$ for every Lie polynomial g. (We are still using the abbreviated notation introduced in 3.2, whereby the basis element x_i is denoted by i.) Let us verify that Δ^* is a contraction of the cochain complex $\Xi^*_{0,*}$. Suppose that $\omega = [\alpha_1, [\alpha_2, .., [\alpha_q, q+1]..]]$ is a basis element with $\alpha_1 \neq 1$. Then $\Delta^*\omega = 0$. On the other hand $\Delta^*\gamma^*_{st}\omega = 0$ for all (s, t) except for

$$\Delta^*\gamma^*_{01}\omega = \Delta^*[1, [\alpha_1 + 1, [\alpha_2 + 1, .., [\alpha_q + 1, q + 2]..]]] = \omega \ ,$$

and therefore $(\Delta^*\partial''^* + \partial''^*\Delta^*)\,\omega = \omega$. Next we must consider the second type of basis element, namely those ω of the form $[1, [\alpha_2, .., [\alpha_q, q+1]..]]$. For these we have still $\Delta^*\gamma^*_{01}\omega = \omega$ and $\Delta^*\gamma^*_{0t}\omega = 0$ for $t > 0$; but now

$$\begin{aligned} \Delta^*\gamma^*_{st}\,\omega &= -\,\gamma^*_{s-1,t-1}\Delta^*\omega \qquad \text{for } 1 < s < t \leq q+2 \\ \Delta^*\gamma^*_{1t}\,\omega &= \Delta^*[[1, t], [\varepsilon_2, [\ldots, [\varepsilon_q, \varepsilon_{q+1}]..]]] \end{aligned}$$

where $\varepsilon_j = \alpha_j$ if $\alpha_j < t$, and $\varepsilon_j = \alpha_j + 1$ if $\alpha_j \geq t$; here α_{q+1} means $q+1$;

$$\begin{aligned} &= \Delta^*[1, [t, [\varepsilon_2, [\ldots, [\varepsilon_q, \varepsilon_{q+1}]..]]] \\ &= [t-1, [\varepsilon_2 - 1, [\ldots, [\varepsilon_q - 1, \varepsilon_{q+1} - 1]..]]] \\ &= -\,\gamma^*_{0,t-1}\Delta^*\omega \qquad \text{for all } 1 < t \leq q+2\,. \end{aligned}$$

Therefore $(\Delta^*\partial''^* + \partial''^*\Delta^*)\,\omega = \omega$ for ω of the second type. This proves that Δ^* contracts $\Xi^*_{0,*}(k)$. Hence its dual Δ is a chain contraction of the column $\Xi_{0,*}(k)$.

The pth column $\Xi_{p,*}(k)$ has $\Xi_{p,q} = \mathrm{Lie}^*_{q+1} \otimes k[\Sigma^p_{q+1}]$. To contract this column we define $\theta : k[\Sigma^p_q] \to k[\Sigma^p_{q+1}]$ by

$$\theta\,[\sigma_1|\sigma_2|\ldots|\sigma_p] = [\bar\sigma_1|\bar\sigma_2|\ldots|\bar\sigma_p]$$

where $\bar\sigma_j$ is the shift of σ_j having $\bar\sigma_j(1) = 1$ and $\bar\sigma_j(t) = \sigma_j(t-1) + 1$ for $t > 1$. This θ has the following commutation relations with the Γ-operators defined in 2.5: first, $c_{01}\theta = 1$ and $c_{0t}\theta = \theta c_{0,t-1}$ for $t > 1$; and second, $c_{st}\theta = \theta\, c_{s-1,t-1}$ for $0 < s < t$. Combining these with the duals of the above formulae for $\Delta^* c^*_{st}$, we find that the homomorphisms

$$(\Delta \otimes \theta) :\ \mathrm{Lie}^*_q \otimes k\big[\Sigma^p_q\big] \longrightarrow \mathrm{Lie}^*_{q+1} \otimes k\big[\Sigma^p_{q+1}\big]$$

give a chain contraction of the pth column. We have now shown that $\Xi(k)$ is an acyclic bicomplex, because the homology of every column vanishes. □

3.4 Proposition. *The homology groups $H\Xi_i(L^{\otimes r})$ of the Γ-modules $L^{\otimes r}$ are as follows.*

$$H\Xi_i(L) \approx \begin{cases} k & \textit{for } i = 0 \\ 0 & \textit{for } i > 0, \end{cases}$$

and $H\Xi_i(L^{\otimes r}) \approx 0$ for all i when $r = 0$ and when $r > 1$.

Proof. In the case $r \geq 1$, the result follows when we apply Lemma 3.2 to calculate $H\Xi_i(L^{\otimes(r-1)} \otimes L)$. The single non-zero value occurs when $r = 1$, when $L^{\otimes(r-1)}$ is the Γ-module with constant value k.

The case $r = 0$ follows from Lemma 3.3, because $L^{\otimes 0}$ is the constant Γ-module with value k. □

3.5 Theorem. *There is a isomorphism $H\Xi_*(F) \approx \mathrm{Tor}^{\Gamma}_*(L^*, F)$, functorial in the left Γ-module F.*

Proof. It is sufficient to establish a natural isomorphism $H\Xi_0(F) \approx L^* \otimes_\Gamma F$, and to prove that $H\Xi_i(P) = 0$ for $i > 0$ when P is projective.

The right Γ-module L^* is identified with the functor t of ([10, 1.4]) in which $t[n]$ is the module of basepoint-preserving maps from $[n]$ to the ring k (where the basepoint of k is 0).

Let Γ_n be the free right Γ-module on one element situated at the object $[n]$. Explicitly, $\Gamma_n[m]$ is the free k-module on $\mathrm{Hom}_\Gamma([m], [n])$, and the Γ-module structure arises from composition. We compute $L^* \otimes_\Gamma F$ from the exact sequence of ([10, 1.4.1])

$$\Gamma_2 \xrightarrow{\alpha} \Gamma_1 \xrightarrow{\beta} L^* \longrightarrow 0$$

in which β maps the generator of Γ_1 to the based map $[1] \to k$ taking 1 to 1, and α maps the generator of Γ_2 to $p_{\{12\}} - p_{\{1\}} - p_{\{2\}}$, where $p_X : [2] \to [1]$ is defined by $p_X^{-1}\{1\} = X$. Tensoring this sequence with F and using the isomorphisms $\Gamma_n \otimes_\Gamma F \approx F[n]$, we deduce that $L^* \otimes_\Gamma F$ is the cokernel of

$$c_{12} - c_{01} - c_{02} : \quad F[2] \longrightarrow F[1] \; . \tag{3.5.1}$$

On the other hand, $H\Xi_0(F)$ is by definition the cokernel of

$$\partial' + \partial'' : \quad \big(\mathrm{Lie}_1^* \otimes k[\Sigma_1] \otimes F[1]\big) \oplus \big(\mathrm{Lie}_2^* \otimes F[2]\big) \longrightarrow \mathrm{Lie}_1^* \otimes F[1] \; .$$

Since Lie_1^* and Lie_2^* are isomorphic to k, and ∂' is zero in this bidegree (from the bar construction on the trivial group Σ_1), this formula for $H\Xi_0(F)$ reduces to (3.5.1). The isomorphism $H\Xi_0(F) \approx L^* \otimes_\Gamma F$ therefore holds.

We now need to show that $H\Xi_i(P) = 0$ for $i > 0$ when P is projective. Since the tensor powers $L^{\otimes r}$ are projective generators of Γ-*mod* (by [10] or [11]), and $H\Xi_*$ commutes with sums, it is enough to prove this for the cases $P = L^{\otimes r}$ for $r \geq 0$. This was done in 3.4, so the theorem is completely proved. □

3.6 Corollary. *The complex $\Xi(k\Gamma)$ is a projective resolution of the right Γ-module L^*.*

Proof. The Γ-bimodule $k\Gamma$ is projective both as a left Γ-module and as a right Γ-module. We have

$$\Xi_{p,q}(k\Gamma) \approx \mathrm{Lie}^*_{q+1} \otimes k\big[\Sigma^p_{q+1}\big] \otimes \Gamma_{q+1} \; .$$

where Γ_{q+1} is the free right Γ-module as in 3.5. Therefore each $\Xi_{p,q}(k\Gamma)$ is projective in *mod*-Γ, as both $k[\Sigma^p_{q+1}]$ and Lie_{q+1} (by 2.1) are k-free. Thus $\Xi(k\Gamma)$ is a projective complex of right Γ-modules; it is augmented over L^* by the following composite involving the map β of 3.5:

$$\Xi_{0,0}(k\Gamma) \;=\; \mathrm{Lie}^*_1 \otimes \Gamma_1 \;\approx\; \Gamma_1 \xrightarrow{\beta} L^* \,.$$

By Theorem 3.5, the homology of $\Xi(k\Gamma)$ is $\mathrm{Tor}^\Gamma_*(L^*, k\Gamma)$. This vanishes in positive degrees, since $k\Gamma$ is a free left module; and β induces an isomorphism of $L^* \otimes_\Gamma k\Gamma$ with L^* by the calculation in the proof of 3.5. Hence the augmented total complex is exact, and $\Xi(k\Gamma)$ is a projective resolution of L^*. □

Using the cyclic property of the Lie operad one can construct analogues of 3.5 and 3.6 for $\mathcal{F}$-modules, where $\mathcal{F}$ is the category of non-empty finite sets (without basepoint). This variant will be treated elsewhere.

3.7 Corollary. *There is a functorial isomorphism* $\pi_*(F) \approx H\Xi_*(F)$ *for* Γ*-modules* F*.*

Proof. This follows immediately from Theorem 3.5 and the natural isomorphism $\mathrm{Tor}^\Gamma_*(L^*, F) \approx \pi_*(F)$ of ([10, 2.2]). □

The bicomplex $\Xi(F)$ is much smaller, and more transparent in its structure, than the Robinson-Whitehouse complex which served the same purpose in [11]. The application which we give in Sect. 5 makes clear that $\Xi(F)$ has precise geometrical origins in the theory of E_∞-structures and infinite delooping.

4. The Γ-homology and Γ-cohomology of graded commutative algebras

Let $A = \{A_n\}_{n\in\mathbb{Z}}$ be an associative graded ring with unit which is commutative in the usual graded sense: that is, $yx = (-1)^{mn}xy$ when $x \in A_m$ and $y \in A_n$. Let B be an A-algebra, and G a B-module. (Here and henceforth the qualification “graded” is usually suppressed, but is to be understood; and unmarked tensor products are over the ground ring A.)

We denote by $(B|A)^\otimes$ the tensor algebra of B over A. Then $(B|A)^\otimes \otimes G$ has a natural Γ-module structure over the ring B: if $\varphi : [n] \to [m]$ is any morphism in Γ, we set

$$\varphi_*(b_1 \otimes \cdots \otimes b_n \otimes g) = \varepsilon\, c_1 \otimes \cdots \otimes c_m \otimes h$$

in which

$$\begin{array}{lll} c_i = b_{i_1} \dots b_{i_r} & \text{if } \varphi^{-1}(i) = \{i_1, \dots, i_r\} & \text{where } i_1 < i_2 < \cdots < i_r \\ h = b_{j_1} \dots b_{j_s} g & \text{if } \varphi^{-1}(0) = \{0, j_1, \dots, j_s\} & \text{where } j_1 < j_2 < \cdots < j_s \end{array}$$

and in which ε is the sign of the permutation that rearranges $\{1, 2, \ldots , n\}$ in the order in which b_1,...,b_n appear in the expansion of the product $c_1 \ldots c_m$. When φ happens to be a permutation σ (regarded as an isomorphism in Γ) this means that φ_* rearranges the factors and multiplies by the sign (compare [10, p. 158]):

$$\sigma_*(b_1 \otimes \cdots \otimes b_n \otimes g) \quad = \quad \varepsilon(\sigma)\, b_{\sigma^{-1}1} \otimes \cdots \otimes b_{\sigma^{-1}n} \otimes g \ .$$

4.1 Definition. *The Γ-cotangent complex of B over A is the bicomplex of B-modules resulting from applying the Ξ-construction of Sect. 2 to the above Γ-module:*

$$\mathcal{K}(B|A) = \Xi((B|A)^{\otimes} \otimes B) \ .$$

The Γ-homology and Γ-cohomology of B relative to A, with coefficients in the B-module G, are defined as the homology and cohomology of $\mathcal{K}(B|A)$:

$$H\Gamma_*(B|A;\ G) = H\Xi_*(\mathcal{K}(B|A) \otimes_B G) \approx H\Xi_*((B|A)^{\otimes} \otimes_A G)$$

$$H\Gamma^*(B|A;\ G) = H\Xi^* \operatorname{Hom}_B(\mathcal{K}(B|A),\ G) \approx H\Xi^* \operatorname{Hom}_A((B|A)^{\otimes},\ G)$$

where the bicomplexes on the right are given the differentials induced from those in the centre column. Since the Γ-modules here are graded, all these constructs have a further internal grading.

Conditions for the homotopy invariance of these definitions are given in 4.5 below.

4.2 Theorem. *For any graded A-algebra B and B-module G*

$$H\Gamma_*(B|A;\ G) \approx \pi_*((B|A)^{\otimes} \otimes G) \ .$$

Proof. This is a special case of 3.7. □

4.3 Corollary. *The above definition of Γ-homology is consistent with the definitions given for ungraded commutative algebras in* [11] *and in* [17].

Proof. In the ungraded case, the Γ-module $\mathcal{K}(B|A) \otimes G$ is the Loday functor of ([9, 3.2]). In [11] and [17] it is proved that the original definitions of Γ-homology give groups isomorphic to the stable homotopy of this functor. □

4.4. Homotopy invariance of the cotangent complex

In applications we often need a homotopy invariance property for the cotangent complex and for Γ-cohomology. By this we mean that the levelwise extensions of these functors to the category of simplicial graded commutative algebras should respect weak equivalences. This is true for the cotangent complex and for Γ-homology provided that B is flat over A. It is true for Γ-cohomology if B is a projective A-module, or if B is flat and $\operatorname{Ext}^s_A(B, G) = 0$

for all $s > 0$, where G is the coefficient module. When these conditions do not hold, a homotopy invariant cotangent complex can be obtained by replacing B by a simplicial projective resolution, then proceeding as above.

General properties of Γ-homology, such as the transitivity exact sequence, are proved in [17]. Some calculations are given in [14] and [15]. In particular, the Γ-homology of a polynomial algebra in one variable is isomorphic to the topological homology of the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$.

5. Application: obstruction theory for E_∞ multiplications on ring spectra

5.1. Multiplicative actions of E_∞ operads on spectra

Let V be a commutative ring spectrum in the sense of classical homotopy theory. Thus there are a given unit map $\eta : S \to V$ and a multiplication $\mu : V \wedge V \to V$ satisfying the homotopy associativity, homotopy commutativity and homotopy unit axioms. We work in a category of spectra which is symmetric monoidal with respect to smash product, such as the category of S-modules [5].

We investigate E_∞ structures on V. An E_∞ structure means an action of an E_∞ operad $\mathcal{C}$ on V. This is prescribed by maps

$$\mu_m : \mathcal{C}_m \ltimes_{\Sigma_m} V^{(m)} \longrightarrow V$$

where $V^{(m)}$ is the mth smash power. The μ_m are required to be compatible with composition (that is, to form a morphism of non-unital operads from $\mathcal{C}$ to the endomorphism operad $\mathrm{End}(V)$); and μ_2 must be the given multiplication μ. In the category of S-modules, such an E_∞ structure guarantees that V is equivalent to a commutative S-algebra: there are no obstructions to higher coherence of the unit η.

We shall take $\mathcal{C}$ to be the E_∞ operad $\mathcal{T}$ introduced in ([17, 1.5]). This operad of cell complexes is the product of the Barratt-Eccles operad with the tree operad. The Barratt-Eccles operad is E_∞: it has mth space $E\Sigma_m$, and composition is induced by wreath product of permutations. The mth space in the tree operad is the space $\tilde{T}_m$ of trees with root and leaves labelled by the elements of $[m] = \{0, 1, \ldots, m\}$, and composition is by grafting trees. As we explained in [17], the reason for our choice is that the product operad $\mathcal{T}$, unlike the Barratt-Eccles operad, is a cofibrant E_∞ operad in the model category structure defined by Hinich [7].

It is useful to note that the operad $\mathcal{T}$ can also be written $\mathcal{T}_m \approx \tilde{E}\Sigma_m \times_{\Sigma_m} \tilde{T}_m$ where $\tilde{E}\Sigma_m$ is the two-sided bar construction on the symmetric group Σ_m, and the action of Σ_m in the operad structure is the left action on $\tilde{E}\Sigma_m$.

We shall now explain how we propose to classify E_∞ ring structures by an inductive procedure.

5.2. The boundary and the diagonal filtration in $\mathcal{T}$

The bar filtration on $E\Sigma_m$ induces a filtration on $\mathcal{T}_m$. Let us denote the rth space in the filtration of $\mathcal{T}_m$ by $\mathcal{T}_m^r$. Thus $\mathcal{T}_m^0 = \Sigma_m \times \tilde{T}_m$, and $\mathcal{T}_m^r = \emptyset$ when $r < 0$.

We shall denote by $\partial\tilde{T}_m$ the subspace of *decomposable* or *fully-grown* trees in $\tilde{T}_m$. These are the trees in which at least one internal edge has length 1. (This is the space denoted by T_m in [17].)

For each j-element subset of $\{1, 2, \ldots, m\}$, the tree space $\tilde{T}_m$ has a *face* homeomorphic to $\tilde{T}_i \times \tilde{T}_j$, when $i, j \geq 2$ and $i + j = m + 1$. It contains just those trees in which an edge of length 1 separates the j leaves labelled by the subset from the $i - 1$ other leaves and the root. The stabilizer of this face is a conjugate of $\Sigma_{i-1} \times \Sigma_j$ in Σ_m. The union of all the faces is $\partial\tilde{T}_m$, and the intersection of two faces is either empty or is a common subface.

In the operad $\mathcal{T}$, we define a *face* of $\mathcal{T}_m$ to be a subspace $E\Sigma_m \times F$, where F is a face of the tree space $\tilde{T}_m$. The *boundary* $\partial\mathcal{T}_m$ of $\mathcal{T}_m$ is $E\Sigma_m \times \partial\tilde{T}_m$, which is the same as the union of all the faces. In any action

$$\mu_m : \mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)} \longrightarrow V$$

of $\mathcal{T}$ on a spectrum V, the restriction of μ_m to any face $(E\Sigma_m \times F) \ltimes V^{(m)}$ is determined, up to coherent homotopy, by the corresponding composite

$$\mu_i \times \mu_j : (\mathcal{T}_i \times \mathcal{T}_j) \ltimes_{\Sigma_{i-1}\times\Sigma_j} V^{(m)} \longrightarrow V\ .$$

This is true because $E\Sigma_m$ has $E\Sigma_{i-1} \times E\Sigma_j$ as a (filtered) $(\Sigma_{i-1}\times\Sigma_j)$-equivariant deformation retract. There is an explicit standard retraction $E\Sigma_m \longrightarrow E\Sigma_{i-1} \times E\Sigma_j$, defined using shuffles, which we always use to extend the above composite $\mu_i \times \mu_j$ over the whole face. These retractions preserve filtration, and are coherent with respect to iterated face inclusions. Hence the restriction of μ_m to $\mathcal{T}_m^r \cap \partial\mathcal{T}_m$ is determined by the restrictions of the $\mu_i \times \mu_j$ to $(\mathcal{T}_i^r \times \mathcal{T}_j^r) \ltimes V^{(m)}$, where $i + j = m + 1$. We now define the *diagonal filtration* ∇ on $\mathcal{T}$ by setting $\nabla^r\mathcal{T}_m = \mathcal{T}_m^{r-m}$; and we define $\partial\nabla^r\mathcal{T}_m$ to be $\nabla^r\mathcal{T}_m \cap \partial\mathcal{T}_m$. It follows that μ_m restricted to $\partial\nabla^r\mathcal{T}$ is determined by the $\mu_i \times \mu_j$ restricted to $\nabla^{r-1}\mathcal{T} \times \nabla^{r-1}\mathcal{T}$, because face inclusions strictly increase the lower index. This permits us to make inductive constructions.

5.3 Definition. *An n-stage for an E_∞ ring structure on V is a sequence of maps*

$$\mu_m : \nabla^n\mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)} \longrightarrow V$$

which on their restricted domain of definition satisfy the requirements for a morphism of operads from $\mathcal{T}$ to $\mathrm{End}(V)$.

A 2-stage for an E_∞ ring structure is nothing but a map $\mu : V \wedge V \to V$. A 3-stage incorporates an associativity homotopy $\mu(1 \wedge \mu) \simeq \mu(\mu \wedge 1)$ and a commutativity homotopy $\mu \simeq \tau\mu$, where τ interchanges factors. This

3-stage extends to a 4-stage if and only if these homotopies satisfy the well-known pentagonal and hexagonal homotopy conditions and the condition that the commutativity homotopy be homotopy commutative.

We denote by $V_*(X)$ and $V^*(X)$ the V-homology and V-cohomology of the spectrum or space X. The multiplication on V gives all the usual products in these theories. We denote by R the graded coefficient ring $V_*(S^0)$, and by Λ the dual Steenrod algebra $V_*(V)$, which is a graded Hopf R-algebroid. We make the standing assumption that Λ is R-flat, and that there is a perfect universal coefficient isomorphism for all n

$$V^*(V^{(n)}) \approx \mathrm{Hom}_R(\Lambda^{\otimes n},\ R)\ .$$

The augmentation $\Lambda \to R$ makes R into a Λ-module, allowing us to define the complex $\mathrm{Hom}_\Lambda(\mathcal{K}(\Lambda|R),\ R)$ and its cohomology $H\Gamma^*(\Lambda|R;\ R)$. The grading on the rings gives a third grading on the bicomplex $\mathcal{K}(\Lambda|R)$. The associated single complex $\mathrm{Tot}\,\mathcal{K}(\Lambda|R)$ and the cohomology $H\Gamma^*(\Lambda|R;\ R)$ are therefore bigraded.

5.4 Proposition. *Let V be a ring spectrum satisfying the above universal coefficient condition. Then the obstruction to extending an n-stage on V to an $(n+1)$-stage is an $(n, 2-n)$-cocycle of the complex* $\mathrm{Tot}\,\mathrm{Hom}_\Lambda(\mathcal{K}(\Lambda|R);\ R)$. *Given a fixed $(n-1)$-stage, the obstruction cocycles to extending different n-stages form a single cohomology class in this complex.*

Proof. In order to extend an n-stage $\{\mu_m\}$ to an $(n+1)$-stage, we require for each integer m in the range $2 \le m \le n+1$ an extension of $\mu_m : \nabla^n\mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)} \longrightarrow V$ over $\nabla^{n+1}\mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)}$. The extension is determined on $\partial\nabla^{n+1}\mathcal{T}_m \ltimes_{\Sigma_m} V^{(m)}$ already by the condition that $\{\mu_m\}$ should respect compositions, as explained in 5.2. The obstruction to extending μ_m therefore lies in the group

$$V^1\Big(\big(\nabla^{n+1}\mathcal{T}_m/\big(\nabla^n\mathcal{T}_m \cup \partial\nabla^{n+1}\mathcal{T}_m\big)\big) \wedge_{\Sigma_m} V^{(m)}\Big)\ .$$

But we have

$$\nabla^{n+1}\mathcal{T}_m/\big(\nabla^n\mathcal{T}_m \cup \partial\nabla^{n+1}\mathcal{T}_m\big) \approx \big(E\Sigma_m^{n-m+1}/E\Sigma_m^{n-m}\big) \wedge (\tilde{T}_m/\partial\tilde{T}_m)$$

where $\partial\tilde{T}_m$ is the space of fully-grown (or decomposable) trees, and Σ_m acts freely upon the factor $E\Sigma_m^{n-m+1}/E\Sigma_m^{n-m}$, which is a wedge of spheres of dimension $n-m+1$, indexed by Σ_m^{n-m+2}. It is known that $\tilde{T}_m/\partial\tilde{T}_m$ has the homotopy type of a wedge of $(m-1)!$ spheres of dimension $m-2$, and that its homology is isomorphic as a Σ_m-module to Lie_m^* with the twisted sign as in 2.1 (see for instance [20]). Thus the quotient $\nabla^{n+1}\mathcal{T}_m/(\nabla^n\mathcal{T}_m \cup \partial\nabla^{n+1}\mathcal{T}_m)$ is a wedge of $(n-1)$-dimensional spheres. From the perfect universal coefficient formula $V^*(V^{(m)}) \approx \mathrm{Hom}_R(\Lambda^{\otimes m},\ R)$, we therefore have obstructions (to the existence of an n-stage) in all the groups

$$\begin{aligned} &V^1\big((\nabla^{n+1}\mathcal{T}_m/(\nabla^n\mathcal{T}_m \cup \partial\nabla^{n+1}\mathcal{T}_m)\big) \wedge_{\Sigma_m} V^{(m)}\big) \\ &\qquad\qquad\approx \mathrm{Hom}_R^{2-n}\left(\mathrm{Lie}_m^* \otimes R\big[\Sigma_m^{n-m+1}\big] \otimes \Lambda^{\otimes m},\ R\right) \\ &\qquad\qquad\approx \mathrm{Hom}_\Lambda^{n-m+1,\, m-1,\, 2-n}(\mathcal{K}(\Lambda|R),\ R) \end{aligned}$$

from the definition of $\mathcal{K}(\Lambda|R)$ in 4.1. This sequence of elements, for m in the range $2 \le m \le n+1$, forms an $(n, 2-n)$-cochain θ of $\mathrm{Tot}\,\mathrm{Hom}_\Lambda(\mathcal{K}(\Lambda|R);\ R)$. (The second grading $2-n$ is the internal grading of the cohomology theory V^*.)

A completely analogous argument shows that the various n-stages which extend a given $(n-1)$-stage are enumerated by difference $(n-1, 2-n)$-cochains of $\mathrm{Tot}\,\mathrm{Hom}_\Lambda(\mathcal{K}(\Lambda|R);\ R)$.

We must now investigate how the obstruction cochain θ changes when the n-stage is varied. This corresponds to analysing the attaching maps connecting the quotients in the diagonal filtration: we show that these correspond to the differentials in the Ξ-complex. Suppose the n-stage is altered by a difference $(n-1, 2-n)$-cochain ρ. The obstruction component $\theta^{n-m+1,\, m-1,\, 2-n}$ is altered by

$$\delta' \rho^{n-m,\, m-1,\, 2-n} + \delta'' \rho^{n-m+1,\, m-2,\, 2-n}$$

where the two terms correspond to the two factors in the smash product decomposition of $\nabla^{n+1}\mathcal{T}_m/(\nabla^n\mathcal{T}_m \cup \partial\nabla^{n+1}\mathcal{T}_m)$ displayed above. In the first of these two terms, δ' is the cohomology differential in the bar construction, arising from the smash factor $E\Sigma_m^{n-m+1}/E\Sigma_m^{n-m}$. In the second term, δ'' is the cohomology dual of the vertical differential in the Ξ-construction (2.5), arising from the formula for the contribution to the boundary from the smash factor $\tilde{T}_m/\partial\tilde{T}_m$. Here we use an explicit isomorphism between the homology of the space $\tilde{T}_m/\partial\tilde{T}_m$ and the representation Lie_m^*, which was established in [20]. The calculation of the boundary is given in the proof of 6.4 of [17], and the evaluation of δ'' then proceeds as in [16].

The above analysis of the coboundary shows that the indeterminacy in θ obtained by altering the n-stage therefore equals the group of $(n, 2-n)$-coboundaries in $\mathrm{Tot}\,\mathrm{Hom}_\Lambda(\mathcal{K}(\Lambda|R);\ R)$. The same analysis shows in the usual way that θ is a cocycle. This completes the proof of the proposition. □

5.5 Theorem. *Let V be a ring spectrum satisfying the universal coefficient condition as in* 5.4. *Suppose given an* $(n-1)$*-stage μ for V which can be extended to an n-stage. Then there is a natural Γ-cohomology class* $[\theta] \in H\Gamma^{n,2-n}(\Lambda|R;\ R)$*, the vanishing of which is necessary and sufficient for μ to be extendable to an* $(n+1)$*-stage.* □

5.6 Theorem. *Let V be as in* 5.5 *above. Suppose V is homotopy commutative and homotopy associative, and that* $H\Gamma^{n,2-n}(\Lambda|R;\ R) = 0$ *for all* $n \ge 4$. *Then V has an E_∞-structure. If further* $H\Gamma^{n,1-n}(\Lambda|R;\ R) = 0$ *for all* $n \ge 3$*, then this E_∞-structure is unique up to homotopy.* □

Goerss and Hopkins have proved a very similar theorem to 5.6 in the course of extensive unpublished work based upon [6]. An important application shows that the Lubin-Tate spectra of [12] have E_∞ structures.

5.7 Corollary (Goerss-Hopkins). *The Lubin-Tate spectrum E_n of a Honda formal group law of height n has an E_∞ multiplicative structure, and this structure is unique up to homotopy.*

Proof. We need the results of the elegant homological treatment of $\Lambda = V_*V$, due to Hopkins and Miller, which is presented in Part 3 of [12]. These arguments show that the perfect universal coefficient theorem holds for $V = E_n$, and that the André-Quillen cohomology $D^*(\Lambda|R;\ R)$ is all zero. Further, they show that $\mathrm{Ext}^s_R(\Lambda, R) = 0$ for $s > 0$, so that the conditions in 4.4 for the homotopy invariance of Γ-cohomology are met. Therefore the dual in cohomology of Richter's Atiyah-Hirzebruch spectral sequence [14] can be applied to show that $H\Gamma^*(\Lambda|R;\ R)$ is zero. □

An alternative proof that the Γ-cohomology of the dual Steenrod algebra for these spectra is zero is given in [15].

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