

## JUVITOP NOTES, FALL 2017

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### 1. ANDY SENGER, 09/21: OVERVIEW

The goal of this semester is to understand Behrens' memoir on the EHP sequence and its interaction with the Goodwillie tower.

The goal is to understand something about unstable homotopy groups of spheres; I'll go through some older approaches. The EHP sequence is a fiber sequence

$$\Omega^2 S^{2n+1} \xrightarrow{P} S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$$

(note this is only a fibration if this is localized at 2). The  $P$  map is related to the Whitehead product. The  $E$  map is the adjoint to the identity map  $\Sigma S^n \simeq S^{n+1} \rightarrow S^{n+1}$ . The James splitting says

$$\Sigma \Omega \Sigma(S^n) \simeq \Sigma \bigvee_{k \geq 0} (S^n)^{\wedge k}$$

and the  $H$  map is  $\Sigma \bigvee_{k \geq 0} (S^n)^{\wedge k} \rightarrow \Sigma S^{2n} \simeq S^{2n+1}$ .

We have a sequence

$$S^0 \xrightarrow{E} \Omega S^1 \xrightarrow{E} \Omega^2 S^2 \rightarrow \dots \xrightarrow{E} \Omega^n S^n \xrightarrow{E} \Omega^{n+1} S^{n+1} \rightarrow \dots \rightarrow \Omega^\infty \Sigma^\infty S^0 \simeq QS^0$$

and we can use the  $H$  maps to turn this into a sequence of principal fibrations

$$\begin{array}{ccccccc} S^0 & \xrightarrow{E} & \Omega S^1 & \xrightarrow{E} & \Omega^2 S^2 & \longrightarrow & \dots & & \Omega^n S^n & \xrightarrow{E} & \Omega^{n+1} S^{n+1} & \longrightarrow & \dots & \longrightarrow & \Omega^\infty \Sigma^\infty S^0 \simeq QS^0 \\ & \nearrow P & \downarrow H & \nearrow P & \downarrow H & & & & \downarrow H & & \downarrow H & & & & & \\ & & \Omega S^3 & & \Omega^2 S^5 & & & & \Omega^n S^{2n+1} & & \Omega^{n+1} S^{2n+3} & & & & & \end{array}$$

and so we get a spectral sequence

$$E_{m,t}^1 \simeq \pi_{t+m+1} S^{2m+1} \implies \pi_t^s.$$

We can truncate the part where  $m \geq n$ , and get something to  $\pi_{n+t}(S^n)$ . If you go in the right order, the unstable stems you need to compute new unstable stems, are ones you already know. So *if* you can compute differentials, then you can inductively compute all unstable homotopy groups of spheres. But we can't do this, because understanding the  $P$  map is hard.

Toda used this to compute the first 20 unstable stems (starting counting at 0, so up to  $\pi_{19}$ ). He used the product structure and Toda brackets. But we're going to do something else.

Using the Snaith splitting of  $\Sigma^\infty \Omega^n \Sigma^n(S^1)$ , you get a map  $\Omega^{n+1} S^{n+1} \rightarrow Q\mathbb{R}P^n$  which sends the EHP filtration to the cellular filtration. We get a map of fiber sequences

$$\begin{array}{ccccc} \Omega^n S^n & \xrightarrow{E} & \Omega^{n+1} S^{n+1} & \longrightarrow & \Omega^{n+1} S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathbb{R}P^{n-1} & \longrightarrow & Q\mathbb{R}P^n & \longrightarrow & QS^n \end{array}$$

which implies you get a map of spectral sequences from the EHP spectral sequence to a (slightly regraded)  $\pi_*^s$ -Atiyah Hirzebruch spectral sequence (the usual  $E_2$  page of the AHSS is now the  $E_1$  page). This is an isomorphism on  $E^1$  where the homotopy groups in  $E_1$  are stable (because  $\Omega^{n+1} S^{2n+1} \rightarrow QS^n$  is just stabilization, so this map will be an isomorphism when it's already stable).

Call the groups you can compute with this range *metastable*. You can compute metastable groups using the AHSS of  $\mathbb{R}P^\infty$ , and this is actually computable. Mahowald, in his little red book, computed quite a few of these.

But wait – there's more! If I can compute a certain localization, I can get everything. Let  $j$  be the fiber of the map  $bo \xrightarrow{\psi} bsp$  (a lift of the Adams operation) be the image of the  $J$  spectrum. The  $\pi_*^s$ -AHSS of  $\mathbb{R}P^\infty$  maps to the  $j_*$ -AHSS of  $\mathbb{R}P^\infty$ . In further work, Mahowald computes this completely. If all you care about is height 1, this is as good as the sphere spectrum. Using this, *and* the fact that  $\Omega^{2n+1} S^{2n+1} \rightarrow Q\mathbb{R}^{2n}$  is an isomorphism on  $v_1$ -periodic homotopy, Mahowald computed unstable  $v_1$ -local homotopy of odd spheres.

We'd like a more systematic version of this that works at higher heights; the Snaith map kind of came out of nowhere. The plan for fitting this into a framework is to use Goodwillie calculus.

Suppose I have a reasonable functor  $F : \text{Top}_* \rightarrow \text{Top}_*$ . Then Goodwillie calculus produces a tower of principal fibrations

$$\begin{array}{ccc} & \vdots & \\ & \downarrow & \\ D_2 F & \longrightarrow & P_2 F \\ & & \downarrow \\ D_1 F & \longrightarrow & P_1 F \\ & & \downarrow \\ & & P_0 F \end{array}$$

called the *Goodwillie tower*. In good cases, this will converge to  $F$ .

Why care? The layers  $D_i$  of the tower have the following nice description:

$$D_i F = \Omega^\infty(\partial_i(F) \wedge X^{\wedge i})_{h\Sigma_i}$$

So  $F$  (which we want to know about) is unstable, and the  $D_i$ 's are stable – it's  $\Omega^\infty$  of some spectrum. Here  $\partial_i(F)$  is a  $\Sigma_i$ -spectrum (in the naïve sense).

We'll take  $F = \mathbb{1}$  and write  $P_i(\mathbb{1})(X) =: P_i X$ . So we have a tower

$$\begin{array}{ccc}
 D_2 X & \longrightarrow & P_2 X \\
 & & \downarrow \\
 \underbrace{D_1 X}_{\simeq \Omega^\infty \Sigma^\infty X} & \longrightarrow & P_1 X \\
 & & \downarrow \\
 & & P_0 X \simeq *
 \end{array}$$

So the tower starts with the stabilization of  $X$  and tries to build up the unstable  $X$  by adding in some further stable things. But what we really care about is spheres, so let's specialize even further and take  $X = S^n$ . Now we have more results:

**Theorem 1.1** (Aronne-Mahowald).

- $D_i S^n \simeq *$  unless  $i = 2^k$ . (Recall we've localized at 2.)  
So the tower converges must faster than you might expect.
- $S^{2n+1} \rightarrow P_{2^k} S^{2n+1}$  is a  $v_k$ -periodic equivalence.  
So if you only care about height  $k$ , you only need to care about the first  $k$  parts of the tower. We've seen this above at height 1, and this says that, at higher heights, there are just finitely more steps you have to take.
- Computed homology of  $D_{2^k} S^n$  and understood the Steenrod algebra action.

**Theorem 1.2** (Aronne-Dwyer).  $D_{2^k} S^n \simeq \Omega^\infty(\Sigma^{\wedge -k} L(k)_n)$  where  $L(k)_n$  is the (stable) summand of  $\Sigma^\infty(B(\mathbb{Z}/2)^k)^{n\bar{p}}$  (Thom space w.r.t. the  $n$  times the reduced real regular representation). This is the Thom space corresponding to the Steinberg idempotent in the group ring of  $GL_k(\mathbb{Z}/2)$ ?

More concretely,  $L(1)_n \simeq \Sigma^\infty \mathbb{R}P^\infty / \mathbb{R}P^n$ .

Consider  $S^1 \rightarrow \Omega^{2n} S^{2n+1}$  (which induces a map of Goodwillie towers) and take the fibers – this will have the same  $v_n$ -periodic homotopy as  $\Omega^{2n} S^{2n+1}$  since  $S^1$  doesn't have any.

**1.1. Behrens' memoir.** Behrens wants to understand how the Goodwillie tower approach interacts with the EHP approach, and it turns out that this interaction is really nontrivial and allows you to compute basically everything through the Toda range.

The EHP sequence comes about through applying the following maps to spheres:

$$\mathbb{1} \xrightarrow{E} \Omega \Sigma \xrightarrow{H} \Omega \Sigma(-)^{\wedge 2}$$

where the second map comes from the James splitting. We always have a sequence of functors like this, but it's not always a fibration; it's a fibration when applied to the 2-local sphere. This induces maps of Goodwillie towers of these functors, which it turns out can be understood relatively easily in terms of the Goodwillie of the identity. This gives rise to fiber sequences

$$\begin{aligned} P_{2m}(S^n) &\rightarrow \Omega P_{2m}(S^{n+1}) \rightarrow \Omega P_m(S^{2n+1}) \\ P_{2m+1}(S^n) &\rightarrow \Omega P^{2m+1}(S^{n+1}) \rightarrow \Omega P_m(S^{2n+1}) \end{aligned}$$

These induce maps on  $D$  layers, which in particular induce the following fiber sequences (“via the 9-lemma”)

$$\Sigma^n L(k-1)_{2n+1} \rightarrow L(k)_n \rightarrow L(k)_{n+1}.$$

These are due to Kuhn and Takeyasu(sp?). These allow us to build  $L(k)_n$  up inductively from the smaller ones.

Behrens uses the fact that the Taylor coefficients  $\partial_i$  of the identity form an operad and rephrases the Arone-Mahowald computation in terms of the fact of these operations. So you can completely understand what's going on with homology.

He defines some “transfinite spectral sequences” – you want to use one spectral sequence to compute the  $E_2$  term of another, and so he combines these spectral sequences into a tower. If you just input the results of one spectral sequence into the next, the extension problems just keep getting worse.

He combines the spectral sequences from the  $L(k)$  fibrations into a “transfinite AHSS” for  $\pi_*^s L(k)_n$ . He inputs this into the Goodwillie tower to get what he calls the “Transfinite Goodwillie spectral sequence” which converges to  $\pi_* S^n$ .

Finally, he puts this into the EHP spectral sequence (the  $E_1$  term of the EHP spectral sequence is just homotopy groups of spheres) to get the “transfinite EHP spectral sequence”. Part of this is to get consistent naming – everything is coming from the transfinite AHSS (TASS) for  $\pi_*^s L(k)_n$ . This allows him to understand what certain maps do, by understanding what they do on the  $E_1$  page of some other spectral sequence.

This gives a method for computing the transfinite Goodwillie spectral sequence (TGSS) and transfinite EHP spectral sequence (TEHPSS) simultaneously, playing them off each other. This comes in several steps:

- (1) Compute some amount of  $\pi_*^s(L(k))$  using the TAHSS, knowledge of the cell structure of  $\mathbb{R}P^\infty$ , and the Steenrod action on  $H_*(L(k))$  to compute some of the stable attaching maps. This is supposed to be the easy part.
- (2) Compute TGSS for  $S^n$  inductively in  $n$ , starting with  $S^1$ .

The differentials in the GSS essentially correspond to understanding the  $H$  map (Hopf invariants)... well, at least, knowledge of the  $H$  map gives GSS differentials. In particular, in the metastable range this can be described completely in terms of the stable homotopy AHSS for  $\mathbb{R}P^\infty$ . All of the ones you can get from  $H$  maps for  $S^1$  come this way. The rest of the GSS differentials can be gotten by knowledge of the output – you already know about  $\pi_* S^1$ . Using  $E$  and  $P$  maps, which induce maps of TGSS, you can push differentials up.

- (3) Simultaneously compute the TEHPSS by lifting differentials from TGSS. (But you have to do this simultaneously because there might be more Hopf invariants you don't understand.)

Behrens uses this to get everything in the Toda range except one differential. You can get more differentials using more ad hoc methods.

## 2. PETER HAINE, 09/27: OVERVIEW OF GOODWILLIE CALCULUS

Plan:

- (1) analogies from calculus and some ideas and motivation
- (2) cubes and  $n$ -excisive functors
- (3) the Taylor tower

**2.1. Goodwillie's motivation.** Goodwillie wanted to study stable pseudo-isotopy theory and Waldhausen  $A$ -theory. Pseudo-isotopy theory is very geometric, and  $A$ -theory is related to parametrized  $h$ -cobordisms. He wasn't, at first, thinking about stable homotopy theory, and these ideas are pretty general.

Idea: in this geometric setting you probably don't have a Postnikov tower, and we want to get something analogous. We want to imitate the following from calculus: if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is smooth, then for  $n \geq 0$ , Taylor's theorem gives an expression  $f(x) = c_0 + \cdots + c_n x^n + u(x)x^{n+1}$  where  $u(x) \in C^\infty(x)$  is a smooth function and the  $n^{\text{th}}$  Taylor polynomial  $p_n := c_0 + \cdots + c_n x^n$  is uniquely characterized by the following two properties:

- (1)  $\deg(p_n) \leq n$
- (2)  $f - p_n$  vanishes to order  $n$  at  $0 \in \mathbb{R}$ .

Suppose we have a functor defined between two homotopical categories, and we want to approximate that using "polynomial functors". We might be working in a setting that is too general to have a Postnikov tower.

I'm going to be working completely homotopically; I'm going to say " $\infty$ -category" but if you don't like that you can substitute a model category. But these things are very general and model-independent: this is in *Higher algebra*, basically just copied from Goodwillie.

Suppose we have  $F : \text{Sp} \rightarrow \text{Sp}$  and we want to approximate  $F$  by "polynomials". Here's a definition expressing what I mean by "polynomial":

**Definition 2.1.** Write  $\text{Poly}^n(\text{Sp}, \text{Sp})$  for the smallest subcategory of  $\text{Fun}(\text{Sp}, \text{Sp})$  that:

- contains  $(-)^{\wedge m}$  for  $0 \leq m \leq n$
- is closed under colimits
- is closed under (de)suspensions (i.e. is homotopically well-behaved).

**Example 2.2.** If you fix  $C_0, \dots, C_n \in \text{Sp}$  then the functor

$$X \mapsto \bigvee_{i=0}^n C_i \wedge X^{\wedge i}$$

is in  $\text{Poly}^n(\text{Sp}, \text{Sp})$ . Not everything looks like this but everything is generated by things that look like this.

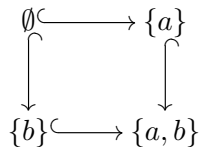
But there are some issues – it’s not super general. The functors  $(-)^{\wedge n}$  preserves filtered colimits, and hence so does everything in  $\text{Poly}^n$ . You should think of this as a smoothness (or finiteness) condition. You might not want this finiteness condition. Furthermore, this definition depends on certain features of spectra. If you want this to be applicable to other contexts, you may not have a monoidal product, or maybe the product and coproduct don’t agree and it’s not clear what to replace  $\bigvee$  with. Finally, there’s no obvious way to check if a functor is in  $\text{Poly}^n$ : you have to find some way to express it as some (de)suspension of a colimit of things like in the example. It’s not even true that the expression is unique.

So Goodwillie realized we should do something different. In the case of spectra,  $\text{Poly}^n$  will be characterized by preserving filtered colimits, plus one more general condition. Here’s a summary of the analogies:

Differential calculus	Functor calculus
Smooth manifold $M$	$\infty$ -category $C$ (nice enough)
smooth map	functor preserving filtered colimits
$x \in M$	$x \in C$
$\mathbb{R}$	$\text{Sp}$
polynomial	excisive functor
subtraction of polynomials	fiber of a map of excisive functors

**2.2. Excisive functors.** Goodwillie’s insight is that degree 1 polynomials should correspond to excisive functors – things that satisfy the usual excision axiom for cohomology theories. One way of writing this is that homotopy pushouts are sent to homotopy pullbacks. Goodwillie realized that this is something you can generalize to higher polynomials by replacing the square in the pushout diagram with a cubical diagram.

**Notation 2.3.** For a finite set  $S \in \text{Fin}$ , write  $P(S)$  for the power set as a poset (this is a  $(\#S + 1)$ -dimensional cube). For example, if  $S = \{a, b\}$  this is



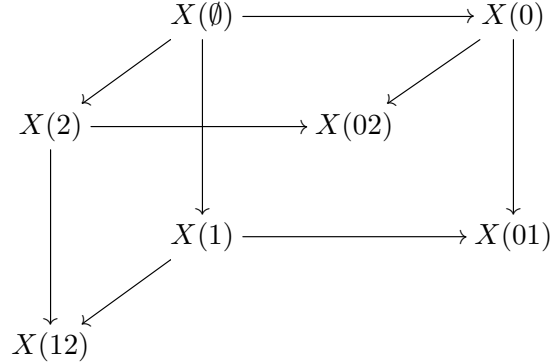
There’s an annoying indexing issue: this is called a 1-cube. Write  $P_{\leq i}(S)$  for the sub-poset of sets of cardinality  $\leq i$ , where  $i \leq \#S$ .

**Definition 2.4.** Let  $C$  be an  $\infty$ -category and  $S \in \text{Fin}$ . Then an  $S$ -cube in  $C$  is a functor  $P(S) \rightarrow C$ . An  $S$ -cube  $X$  is *Cartesian* if

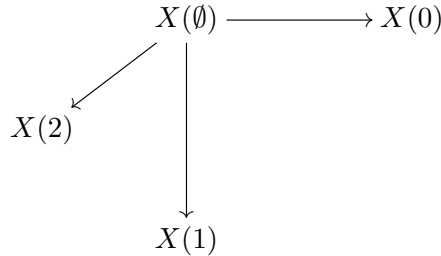
$$X(\emptyset) \rightarrow \lim_{\emptyset \neq S' \subset S} X(S')$$

is an equivalence. (If  $S$  were a square this would be a pullback diagram. In general, you're looking at the limit diagram gotten from removing  $\emptyset$  from the cube.)

Saying a cubical diagram is a pushout is asking for



to be a pushout. But I can ask for a stronger property – I should be able to build the entire cube by taking iterated pushouts of



**Definition 2.5.** An  $S$ -cube is *strongly cocartesian* if  $X$  is a left Kan-extension of  $X|_{P_{\leq 1}(S)}$ .

**Notation 2.6.** An  $n$ -cube is an  $\{0, \dots, n\}$ -cube. (This is sort of off by 1.)

**Definition 2.7.** Let  $C \in \text{Cat}_{\infty}$  admit finite colimits,  $D \in \text{Cat}_{\infty}$ , and  $n$  be a non-negative integer. A functor  $F : C \rightarrow D$  is  *$n$ -excisive* if

$$F(\text{strongly cocartesian } n\text{-cube}) = \text{cartesian } n\text{-cube}.$$

**Example 2.8.**

- A 0-cube in  $C$  is a morphism in  $C$ . Every 0-cube is strongly cocartesian. However, not every zero-cube is cartesian: that means that  $X(\emptyset)$  is the limit of the diagram  $X(0)$ . In order for this to be cartesian, I need  $X(\emptyset) \rightarrow X(0)$  to be an equivalence. In general, being 0-excisive is equivalent to factoring through  $D^{\simeq}$ .
- When  $n = 1$ , being strongly cocartesian is the same as being cocartesian. This agrees with the usual terminology: being 1-excisive means that pushouts get sent to pullbacks.

Warning: the identity functor  $1_C : C \rightarrow C$  is generally *not* 1-exciseive. This is weird: 1-exciseive is supposed to be our analogue of being a linear functor.

**Proposition 2.9.** *A functor  $F : \text{Sp} \rightarrow \text{Sp}$  is polynomial of degree  $n$  iff  $F$  is  $n$ -exciseive and preserves filtered colimits.*

If a functor is  $n$ -exciseive then it is  $m$ -exciseive for  $m \geq n$ .

**Goal 2.10.** To a (reasonable) functor  $F : C \rightarrow D$  we want to associate a tower  $F \rightarrow \dots \rightarrow P_n F \rightarrow \dots \rightarrow P_0 F$ , where  $P_n F$  is  $n$ -exciseive. It will be different from a Postnikov tower in a lot of ways – e.g. we won't expect it to converge, just like we don't expect Taylor series to converge to the function unless the function is analytic.

It is easy to see the following: if we want a universally-defined exciseive approximation to a functor, that would be a left adjoint to the inclusion  $\text{Exc}^n(C, D) \hookrightarrow \text{Fun}(C, D)$ . For example,  $P_0(F)$  is the constant at  $F(*_C)$ , where  $*_C$  is the final object in  $C$ . (You always have a natural transformation  $F \rightarrow P_0 F$  and this is universal.)

How might you construct  $P_1 : \text{Fun}(C, D) \rightarrow \text{Exc}^1(C, D)$ ? I'll give the construction and then say what conditions were needed later. Even as a homotopy theorist, you might want to work in various settings, e.g. spaces over a fixed space, etc. So I'm stating this in some generality.

If  $F : C \rightarrow D$  is 1-exciseive (and all the things I'm about to say are defined), then for all  $x \in C$  I can take the pushout square defining the suspension

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma_C X \end{array}$$

Then  $F(\text{this})$  is supposed to be a pullback square:

$$\begin{array}{ccc} F(X) & \longrightarrow & F(*) \\ \downarrow & & \downarrow \\ F(*) & \longrightarrow & F(\Sigma_C X) \end{array}$$

In the special case where  $F(*) \simeq *$ , then  $F(X) = \Omega_D F(\Sigma_C X)$ . If you want to force  $F$  to be 1-exciseive, you'll have to force  $F(X) = \Omega_D F(\Sigma_C X)$ .

Define  $T_1(F)(X)$  to be the pullback of

$$\begin{array}{ccc} & & F(*) \\ & & \downarrow \\ F(*) & \longrightarrow & F(\Sigma_C X) \end{array}$$

Even if the functor weren't 1-exciseive, you still get a natural transformation  $F \rightarrow T_1(F) \simeq \Omega F \Sigma$ . Define

$$P_1(F) = \text{colim}(F \rightarrow T_1(F) \rightarrow T_1^2(F) \rightarrow \dots).$$



In this special case (where  $F(*) \simeq *$ ), this is  $\simeq \operatorname{colim}_m \Omega^m F \Sigma^m$ . By thinking about the Eilenberg-Steenrod axioms, it turns out that this is enough – this is the first excisive approximation.

**Definition 2.11.** A category  $D \in \operatorname{Cat}_\infty$  is called *differentiable* if:

- (1)  $D$  has finite limits;
- (2)  $D$  has colimits of sequences;
- (3) these commute.

Examples:  $\operatorname{Sp}$ , Spaces (and also for any stable category and for every topos).

We want the excisive approximation to preserve finite limits, but I won't get into the construction.

**Theorem 2.12** (Goodwillie). *Given  $C \in \operatorname{Cat}_\infty$  with finite colimits and a final object,  $D$  a differentiable category and  $n \geq 0$ ,*

- (1) *the inclusion  $\operatorname{Exc}^n(C, D) \hookrightarrow \operatorname{Fun}(C, D)$  has a left adjoint  $P_n$ , and*
- (2)  *$P_n$  preserves finite limits.*

The main example we care about is the following.

**Example 2.13.** Let  $C = D = \operatorname{Spaces}$  and  $F = \mathbb{1}$ . This certainly preserves  $*$ ! Then  $P_1(\mathbb{1})(S^0) = \operatorname{colim}_m \Omega^m \Sigma^m S^0$ . We like this, because if you take its homotopy groups you get homotopy groups of spheres. You get a tower of principal fibrations

$$\cdots \rightarrow P_2(\mathbb{1})(S^0) \rightarrow P_1(\mathbb{1})(S^0) \rightarrow P_0(\mathbb{1})(S^0)$$

that starts with stable homotopy theory and goes to something else. The “something else” is of interest.

### 3. SANATH DEVALAPARKUR, OCTOBER 4: EXAMPLES OF GOODWILLIE CALCULUS

Don't read this; read Sanath's notes instead:

<http://www.mit.edu/~sanathd/goodwillie.pdf>

$\mathcal{C}$  will be an  $\infty$ -category with finite colimits and a terminal object, and  $\mathcal{D}$  is a “differentiable  $\infty$ -category”, which means it has finite limits, sequential colimits, and these commute.

Last time, we saw the category of  $n$ -excisive functors  $\operatorname{Exc}^n(\mathcal{C}, \mathcal{D})$ , and saw that there is a left adjoint  $P_n$  to the inclusion  $\operatorname{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ . If a functor is  $(n-1)$ -excisive, it is  $n$ -excisive. By the definition, we get  $P_n P_{n+k} \simeq P_n$  for  $k \geq 0$ . (Truncating a degree- $(n+k)$  Taylor approximation gives a degree- $n$  Taylor approximation.)

Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. We have a tower  $\cdots \rightarrow P_n F(X) \rightarrow \cdots \rightarrow P_0 F(X)$ .

In the analogy, if we have  $f : \mathbb{R} \rightarrow \mathbb{R}$  then  $p_n f - p_{n-1} f = \frac{f^{(n)}(0)}{n!} x^n$ . The analogous thing to taking differences is taking fibers. So we'll look at the fiber sequence

$$\begin{array}{ccc} D_n F(X) & \longrightarrow & P_n F(X) \\ & & \downarrow \\ & & P_{n-1} F(X) \end{array}$$

If  $\mathcal{C}$  and  $\mathcal{D}$  are things like  $\mathrm{Sp}$  or  $\mathrm{Top}$ , you can apply  $\pi_*$  and get long exact sequences and an exact couple. You get a spectral sequence

$$E_{p,q}^1 = \pi_q D_p F(X) \implies \pi_{p+q} P_\infty F(X)$$

where  $P_\infty$  is the limit of the tower. This won't necessarily converge to  $F$  unless it is *analytic*. Goodwillie has a definition of this, but all I'll say is that  $F$  is "analytic" if  $F$  "agrees" with  $P_k F$  for  $k \gg 0$ . (But there are a bunch of technical conditions I didn't state.)

It would be nice if  $D_n F$  were homogeneous of degree  $n$ .

**Lemma 3.1.**  $P_n D_n F \simeq D_n F$  and  $P_{n-1} D_n F \simeq *$

This is saying that  $D_n F$  is  $n$ -excisive, and the second condition says that  $D_n F$  is homogeneous of degree  $n$ .

*Proof.*  $P_n$  is a left adjoint. Apply  $P_n$  to the cofiber sequence  $D_n F \rightarrow P_n F \rightarrow P_{n-1} F$ . Nothing happens to the second and third terms, so  $P_n D_n F \simeq D_n F$ . For the second statement, apply  $P_{n-1}$  to the fiber sequence. Then the second and third terms are equivalent, so  $P_{n-1} D_n F \simeq *$ .  $\square$

We want to get a closer analogue of the form  $\frac{f^{(n)}(0)}{n!}$ .

**Theorem 3.2.** *Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  is homogeneous of degree  $n$ , and suppose  $\mathcal{D}$  is a stable  $\infty$ -category. Then there is a  $\otimes$   $n$ -linear functor  $LF : \mathcal{C}^n \rightarrow \mathcal{D}$  and*

$$LF(X, \dots, X)_{h\Sigma_n} \simeq F(X).$$

Assume  $\mathcal{C} = \mathrm{Sp}$ . Then show

$$LD_n F(X, \dots, X)_{h\Sigma_n} = (LF(S, \dots, S) \wedge X^{\wedge n})_{h\Sigma_n}$$

and by the theorem the LHS is  $D_n F(X)$ . Think of this as the  $n^{\mathrm{th}}$  derivative; dividing by  $n!$  is the analogue of homotopy orbits.

Let  $\mathcal{C}$  be a pointed  $\infty$ -category with finite limits. Then

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}_*^1(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C}).$$

The idea is that an excisive functor  $F : \mathrm{Top}_*^{\mathrm{fin}} \rightarrow \mathcal{C}$  corresponds to  $\{F(S^n)\}_{n \geq 0}$ . This is the perspective that Jacob Lurie takes in *Higher algebra*.

Suppose we have a functor  $F : \mathcal{C} \rightarrow \mathrm{Sp}(\mathcal{D})$ . There is a functor  $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \rightarrow \mathcal{D}$  and so the composite is  $\mathcal{C} \rightarrow \mathcal{D}$ . Under the above identification we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathrm{Sp}(\mathcal{D}) \simeq \mathrm{Exc}_*^1(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C}) \\ & \searrow & \downarrow \Omega^\infty \\ & & \mathcal{D} \end{array} \quad \begin{array}{c} \swarrow \mathrm{ev}_{S^0} \\ \end{array}$$

In general,  $P_n(\mathrm{ev}_X \circ F) \simeq \mathrm{ev}_X \circ P_n(F)$ . So  $F$  is  $n$ -excisive iff  $\mathrm{ev}_X \circ F$  is  $n$ -excisive for all  $X$ .

$$\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})).$$

The idea is that ‘‘homogeneity is essentially a stable phenomenon’’.

As a consequence, for  $F : \mathcal{C} \rightarrow \mathcal{D}$  there is  $\widetilde{D}_n F : \mathcal{C} \rightarrow \mathrm{Sp}(\mathcal{D})$  such that  $D_n F \simeq \Omega^\infty \circ \widetilde{D}_n F$ .

I’ll show how to deduce most of the Snaith splitting theorem from this story.

Let  $X$  be a space. Then  $\pi_*^s X \simeq \pi_* \Omega^\infty \Sigma^\infty X$ . In general,  $\Omega^n \Sigma^n X$  is hard. It is a theorem of James (related to the James splitting theorem) that

$$\Omega \Sigma X \simeq \left( \bigsqcup_{n \geq 0} X^n \right) / \sim .$$

In his work on delooping, May showed

$$\Omega^n \Sigma^n X \simeq \left( \bigsqcup_{i \geq 0} C^{(n)}(i) \times_{\Sigma_i} X^i \right) / \sim$$

where  $C^{(n)}(i)$  is the space of embeddings  $\sqcup_i I^n \hookrightarrow I^n$  appearing in the little discs operad. You can stop the disjoint union at some level and get a filtration of  $C_n X := \Omega^n \Sigma^n X$ .

**Theorem 3.3** (Snaith).

$$\Sigma^\infty \Omega^n \Sigma^n X \simeq \bigvee_{k \geq 1} \Sigma^\infty (C^{(n)}(k)_+ \wedge X^{\wedge k})_{h\Sigma_k}$$

If you let  $k \rightarrow \infty$ ,  $C^{(k)} = E\Sigma_k$ . As  $n \rightarrow \infty$ , you get

$$\Sigma^\infty \Omega^\infty \Sigma^\infty X \simeq \bigvee_{k \geq 1} \Sigma^\infty X_{h\Sigma_k}^{\wedge k} .$$

We need to define a functor such that applying it to some nice space gets the LHS in the theorem. Let  $K$  be a finite CW complex. Define a functor  $F : \mathrm{Top}_* \rightarrow \mathrm{Sp}$  via  $X \mapsto \Sigma^\infty \mathrm{Top}_*(K, X)$ .

**Theorem 3.4** (Goodwillie). *F is analytic.*

So after imposing connectivity conditions, the associated spectral sequence will strongly converge.

Let  $\mathcal{E}$  be the category of finite sets  $\underline{n} = \{1, \dots, n\}$  and surjections. Inside this is the full subcategory  $\mathcal{E}_d \subset \mathcal{E}$  consisting of  $\underline{n}$  such that  $n \leq d$ . If  $X$  is a space, define  $X^\wedge : \mathcal{E}^{op} \rightarrow \mathrm{Top}_*$  sending  $\underline{n} \mapsto X^{\wedge n}$ .

Fix  $K$  throughout; it will not appear in the notation.

**Theorem 3.5** (Arone).

$$P_d F(X) = \text{Map}_{\text{Fun}(\mathcal{E}_d^{\text{op}}, \text{Sp})}(\Sigma^\infty K^\wedge, \Sigma^\infty X^\wedge)$$

The natural transformation  $F \rightarrow P_d F$  takes  $f : K \rightarrow X$  to  $\Sigma^\infty \circ f^\wedge : \Sigma^\infty \circ K^\wedge \rightarrow \Sigma^\infty \circ X^\wedge$ .

If  $K = S^n$  and  $X = \Sigma^n Y$  for some  $Y$ , then the tower splits, so

$$\Sigma^\infty \Omega^n \Sigma^n Y = F(\Sigma^n Y) \simeq \bigvee_{d \geq 0} D_d F(\Sigma^n Y).$$

(The bottom cell of  $Y$  is larger than the radius of convergence of the functor, and  $Y$  here is  $X$  in the theorem.) So the goal is to identify  $D_d F(\Sigma^n Y)$ .

Arone's theorem gives a homotopy pullback square

$$\begin{array}{ccc} P_d F(X) & \longrightarrow & \text{Map}_{\text{Sp}}(\Sigma^\infty K^{\wedge d}, \Sigma^\infty X^{\wedge d})_{h\Sigma_d} \\ \downarrow & & \downarrow \\ P_{d-1} F(X) & \longrightarrow & \text{Map}_{\text{Sp}}(\Sigma^\infty \delta_d K, \Sigma^\infty X^{\wedge d})_{h\Sigma_d} \end{array}$$

where  $\delta_d K$  is the “fat diagonal”  $\{(x_1, \dots, x_d) : \exists i \neq j \text{ with } x_i = x_j\}$ .

So

$$\begin{aligned} D_d F(X) &= \text{Maps}_{\text{Sp}}(\underbrace{\Sigma^\infty K^{\wedge d} / \delta_d K}_{=: K^{(d)}}, \Sigma^\infty X^{\wedge d})_{h\Sigma_d} \\ &\simeq (\mathbb{D}\Sigma^\infty K^{(d)} \wedge \Sigma^\infty X^{\wedge d})_{h\Sigma_d} \qquad \text{Spanier-Whitehead} \end{aligned}$$

We want to understand this where  $K = S^n$  because that's where the Snaith splitting comes from.

**Goal:** understand  $\mathbb{D}S^n$ . An element  $c \in C^{(n)}(d)$  is a map  $\bigsqcup^d I^n \hookrightarrow I^n$ . Apply the Pontryagin-Thom collapse map to get  $S^n \rightarrow \bigvee^d S^n$ . This begets  $C^{(n)}(d) \xrightarrow{\alpha^{(n,d)}} \text{Map}_*(S^n, \bigvee^d S^n)$ . The adjoint of  $\alpha^{(n,1)}$  is  $\delta^{(n,1)} : C^{(n)}(1) \wedge S^n \rightarrow S^n$ . Overall, we get  $\delta^{(n,1)^{\wedge d}} : C^{(n)}(d)_+ \wedge S^{nd} \hookrightarrow C^{(n)} \wedge S^{nd}$ . It turns out that this factors through the fat diagonal, i.e. we get  $C^{(n)}(d)_+ \wedge S^{n(d)} \rightarrow S^{nd}$ .

So

$$\mathbb{D}\Sigma^\infty S^{n(d)} = \Sigma^{nd} \Sigma_+^\infty C^{(n)}(d).$$

With  $K = S^n$ , this implies that

$$D_d F(\Sigma^{-n} X) \simeq \Sigma^\infty (\mathbb{D}\Sigma^\infty S^{n(d)} \wedge \Sigma^\infty (\Sigma^{-n} X)^{\wedge d})_{h\Sigma_d}$$

which is what we wanted.

There's a way to deduce something very similar to the Kan-Priddy splitting theorem this way. Consider the functor  $F : \text{Sp} \rightarrow \text{Sp}$  sending  $X \mapsto \Sigma^\infty \Omega^\infty X$ . Let  $X_n$  denote the  $n^{\text{th}}$  space of  $X$ .

Then  $\Omega^n X_n = \Omega^\infty X$ , and  $\text{colim}_n \Sigma^{-n} \Sigma^\infty X_n \rightarrow X$  is an equivalence. So

$$\text{colim}_n \Sigma^{-n} F^{S^n}(X_n) \rightarrow F(X)$$

is an equivalence. (I think these  $F$ 's are different: the one on the left is some kind of loop space.) The Goodwillie tower for  $F$  is

$$\begin{array}{ccc} \Sigma^\infty \Omega^\infty X & \longrightarrow & P_2 F(X) \\ & & \downarrow \\ & & P_1 F(X) = X \end{array}$$

Now apply  $\Omega^\infty$  to get

$$\begin{array}{ccc} \Omega^\infty F(X) = \Omega^\infty \Sigma^\infty \Omega^\infty X & \longrightarrow & \Omega^\infty P_2 F(X) \\ & \searrow \mathbb{1} & \downarrow \text{splits} \\ & & \Omega^\infty P_1 F(X) = \Omega^\infty X \end{array}$$

It turns out that when  $X = S^{-1}$ ,  $\Omega^\infty P_2 F(S^{-1}) = \Omega Q \mathbb{R}P_+^\infty \rightarrow \Omega Q S^0$  which splits. This is one loop away from being the actual Kan-Priddy theorem.

#### 4. HOOD CHATHAM, 10/11: THE GOODWILLIE TOWER OF THE IDENTITY FUNCTOR

Everything will be  $p$ -completed. The first paper about this is Brenda Johnson's paper.

We're interested in  $(D_n \mathbb{1})(X) =: D_n(X)$ , intended as a spectrum, not a space (these have different homologies). We know

$$D_n X \simeq \partial_n \wedge_{\Sigma_n} X^{\wedge n}.$$

One way to get at  $\partial_n$  is by looking at the homogenization of the cross-effects functor  $\chi_n$ :

$$\partial_n = \Omega^{i\bar{p}} D_{1,\dots,j}(\chi_n \mathbb{1})(S^i, \dots, S^i)$$

(These are the spaces in the  $\Omega$ -spectrum for  $\partial_n$ .) We haven't defined  $\chi_n$ , but it's pretty explicit – it involves some pullback cube.

There's a map  $\chi_n \mathbb{1} \rightarrow$  some easier multilinear  $\Sigma_n$ -functor. The LHS has the right asymptotic connectivity to be an equivalence from the multilinearization of the LHS to the RHS. This expresses

$$\partial_n \simeq \mathbb{D}P_n$$

where  $\mathbb{D}$  means Spanier-Whitehead dual.  $P_n$  has a cover by contractible sets such that each intersection is either empty or contractible (think Čech filtration). That means the homotopy type of this is the nerve of some poset, where the poset is the thing that gives the combinatorics of this cover. This is completely explicit.

What are the combinatorics of the cover? Given a set  $A$ , the partition complex  $\text{Part}(A)$  is the poset of equivalence relations on  $A$ , partially ordered by refinement (putting two sub-bins into one bin). This poset has a minimum and maximum element, and is contractible. You get

$$P_n := \frac{N(\text{Part}([n]))}{N^+ \cup N^-}$$

where  $N_1^+$  is the poset of partitions where you delete the bottom element. The  $i$ -simplices are chains of length  $i + 1$  that start at the bottom and end at the top. So the quotient is the unreduced suspension of the union on the bottom, and so it's the double suspension of the intersection.

You can make a tree out of this.

Ching shows that the bar construction on an operad is given by certain trees with labeled leaves, and internal nodes are labeled by elements of the operad. Our trees are unlabelled (i.e. labels in a set of size one). So

$$P_n \simeq (B \text{Comm})(n).$$

So  $\partial_n \simeq \mathbb{D}P_n$  is Koszul dual to  $\text{Comm}$ , and so it's a shifted Lie operad.

Also  $\bigvee D_n X$  is a free  $\partial_*$ -algebra on  $X$ , and  $H_* D_1 X = H_* \Sigma^\infty X = H_* X$ .

So we expect  $H_* D_n X$  to be generated by  $H_* D_1 X$  under  $\partial_*$  homology operations. These were identified (at least at  $2^?$ ) by Behrens and Camarena. There is a Lie bracket of degree  $-1$

$$H_i D_m X \otimes H_j D_n X \rightarrow H_{i+j-1} D_{m+n} X.$$

At  $p > 2$  (where the theorem doesn't work anyway...) there's also

$$\beta^\varepsilon \tilde{Q}^j : H_i D_m X \rightarrow H_{i+j-1-\varepsilon} D_{pm} X,$$

Koszul dual to a Dyer-Lashof operation. There is an Adem relation on  $(\beta^\varepsilon \tilde{Q}^j)(\beta^{\varepsilon'} \tilde{Q}^{j'})$  exactly when  $(\beta^\varepsilon Q^j)(\beta^{\varepsilon'} Q^{j'})$  is admissible (i.e. doesn't have an Adem relation).

If you apply this to an odd sphere, the bracket vanishes because of graded anti-commutativity, and

$$H_* D_{p^k} S^{2d+1} \cong \mathbb{F}_p \{ \beta^{\varepsilon_1} \tilde{Q}^{i_1} \dots \beta^{\varepsilon_k} \tilde{Q}^{i_k} u \}.$$

This generator is in degree  $(2d + 1, 1)$ .

If I have an even sphere  $S^{2d}$ ,  $H_* D_* S^{2d}$  is also freely generated under the  $\beta^\varepsilon \tilde{Q}^j$  operations by  $u$  and  $[u, u]$  (but all the other possible brackets are zero).

A theorem of Arone and Dwyer (which will be talked about next time) is that

$$D_n(S^{2i+1}) \simeq \Sigma^? L(n)$$

where  $L(n)$  will be defined later. This gives an alternate computation of  $H_* D_n S^{2d+1}$ , via Kuhn's Whitehead conjecture. But we'll talk about Arone-Mahowald's computation of  $H_* D_n S^{2d+1}$ .

From the filtration of  $P_n$  (which comes from the fact it's the nerve of this partition poset  $\text{Part}(n)$ ), we get a spectral sequence. In particular, we have

$$\begin{aligned} D_n X &\simeq \partial_n \wedge_{\Sigma_n} \Sigma X^{\wedge n} \\ &\simeq \text{Map}_*(P_{n+}, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \end{aligned}$$

and this is what we're going to filter to get a spectral sequence

$$\begin{aligned} E_1^{s,t} &= H_s(\text{Map}(\Sigma^t P_n(t), \Sigma^\infty X^{\wedge n})_{h\Sigma_n}) \\ &= H_s(\text{Map}(P_n(t), \Sigma^\infty X^{\wedge n})_{h\Sigma_n}) \implies H_s D_n X \end{aligned}$$

Here  $P_n(t)$  means the discrete set of  $t$ -simplices, and the suspension of this is the fiber of the map where I pull back the skeleta. This collapses at  $E_2$ .

$i$ -simplices are chains of partitions, and there's a  $\Sigma$ -action by permuting the things you're partitioning. You can break this up into orbits.

$$\text{Map}(P_n(i), \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq \bigvee_{\Lambda} \Sigma^\infty X_{h\Sigma_\Lambda}^{\wedge n}$$

where  $\Lambda$  contains orbits?

The coboundary maps  $\partial_0$  and  $\partial_i$  are zero. For the inner  $\partial_j$ : the face maps forget one partition in the chain. They give maps

$$X_{h\Sigma_\Lambda}^{\wedge n} \rightarrow X_{h\Sigma_{\Lambda'}}^{\wedge n}$$

which are transfers. ( $\Lambda'$  correspond to adding an extra thing to the partition?)

**Definition 4.1.**  $\Lambda$  is “pure” if  $\Sigma_\Lambda \cong \Sigma_{p^{k_0}} \wr \Sigma_{p^{k_1}} \wr \dots \wr \Sigma_{p^{k_n}}$ .

The point is that  $H_*(X_{h\Sigma_p}^{\wedge p})$  contains  $Q^i u$  (where  $u$  is the fundamental class), and  $H_*(X_{h\Sigma_{p^k}}^{\wedge p^k})$  contains  $Q^{i_0} Q^{i_1} \dots Q^{i_k} u$ , and wreath products play well with this. You can generate stuff using multiplication and Dyer-Lashof operations, and the wreath products correspond to just doing Dyer-Lashof operations.

**Definition/ Claim 4.2.**  $P^\bullet \subset C^\bullet$  is a subcomplex where  $P^\bullet$  just consists of pure elements.

We're going to show that this is a quasi-isomorphism, but for odd spheres only.

“Dyer-Lashof operations are detected by elementary abelian groups.” Let  $A_k$  be  $C_p^k$ . The map  $H_*(BA_{k+} \wedge X) \rightarrow H_*(X_{h\Sigma_{p^k}}^{\wedge p^k})$  (gotten by the diagonal on  $X$  and  $A_k \rightarrow \Sigma_{p^k}$ ) sends  $Q_1 \otimes \dots \otimes Q_{i_{k-1}} \otimes Q_{i_k} \otimes u \mapsto Q_{i_1} \dots Q_{i_k} u$ . The point is that this is just hitting pure things, not products of things.

We need to understand  $\text{Tr}_{\Sigma_{\Lambda'}}^{\Sigma_\Lambda} u$  where  $u$  is pure. If  $\Lambda'$  is not a pure orbit, I claim this is zero.

Idea:  $u = \text{Res}_{A_k}^{\Sigma_\Lambda} v$ . Apply the double coset formula:

$$\text{Tr}_K^G \text{Res}_H^G(u) = \sum_{K \times_i H} \text{Res}_{H \times_i \cap K} \text{Tr}_{H \times_i \cap K}^{H \times_i}(u).$$

The transfer from any elementary abelian group to a proper subgroup is always zero. This hits pure stuff, and so we've proven the theorem.

Let  $I^\bullet$  be the quotient of  $P^\bullet \rightarrow C^\bullet$ .

**Claim 4.3.**  $I^\bullet \simeq *$  for  $X = S^{2d+1}$

**Corollary 4.4.**  $E_2 \cong H^*(P^\bullet)$

This is only valued in one column, so we just have to compute that and then the spectral sequence collapses.

**Corollary 4.5.**  $D_n S^{2d+1} \simeq *$  when  $n$  is not  $p^k$

**Theorem 4.6.** Let  $CU^*$  (completely unadmissible) be

$$\mathbb{F}_p\{(\beta^{\varepsilon_1} Q^{s_1}) \wr \dots \wr (\beta^{\varepsilon_k} Q^{s_k}) : s_k > |u|, s_i > ps_{i+1} - \varepsilon_{i+1}\}.$$

Then

$$H^* P^\bullet \cong CU^* \cong \Sigma^k H_* D_{p^k} S^{2d+1}.$$

As before,  $u$  is the fundamental class.

The top dimension corresponds to the finest partitions (longest length), and this corresponds to the isotropy being a wreath product  $\Sigma_p \wr \Sigma_p \wr \dots \wr \Sigma_p$ . As I go down the poset, I'm putting more of these together, and this forces actual relations to hold. When I had isotropy with  $\Sigma_{p^2}$  in one of these spots, then relations would hold, and the only thing that could sit there is admissible things. Differentials should take admissible things to admissible things, and kill everything where you have two admissible things together.

## 5. ROBERT BURKLUND, 10/18: LAYERS IN THE GOODWILLIE TOWER OF THE IDENTITY

Everything is going to be  $p$ -local (this is different from how everything was 2-local, before and after this, though this is easier when  $p = 2$ ). Last time we had the partition complex  $P_n$ , thought of as a poset (excluding the top and bottom element). Notation:

- Let  $\mathcal{P}$  be the collection of subgroups  $\Sigma_{i_1} \times \dots \times \Sigma_{i_k} \hookrightarrow \Sigma_n$ .
- Let  $\mathcal{E}$  be the collection of nontrivial elementary  $p$ -abelian subgroups.
- Let  $\mathcal{E}'$  be the collection of *nontransitive* nontrivial elementary  $p$ -abelian subgroups ( $G = \Sigma_n$ ).
- Let  $\mathcal{F}$  be the collection of all nontransitive subgroups.
- Let  $\mathcal{F}^\circ = \mathcal{F} \setminus \{e\}$ .
- $\ell$  is odd.
- $n = p^k$  (for some fixed  $k$ )
- $\Delta = \mathbb{F}_p^k$
- $\text{Aff}_k := GL_k(\mathbb{F}_p) \rtimes \Delta \subset \Sigma_n$ . This is the normalizer of  $\Delta$  in  $\Sigma_n$ .

**Theorem 5.1.**  $D_n(S^\ell) \simeq \Sigma^{\ell-k} L(k)_\ell$

I haven't defined  $L(k)_\ell$  yet – I'll define it to be whatever we get in the last step. (Take  $\ell$  copies of the reduced regular representation  $\rho$  and take homotopy orbits w.r.t.  $\Delta$ , and take the part of this associated to the Steinberg idempotent.)



*Proof.* Let  $\diamond$  denote unreduced suspension.

$$\begin{aligned}
\Sigma^k D_n(S^\ell) &\simeq \Sigma^k \text{Map}_*(S^1 \wedge P_n^\diamond, \Sigma^\infty S^{\ell\rho})_{h\Sigma_n} \simeq \Sigma^{k-1}(\mathbb{D}P_n^\diamond \wedge \Sigma^\infty S^{\ell\rho})_{h\Sigma_n} \\
&\stackrel{(1)}{\simeq} \Sigma^{k-1}(\mathbb{D}T_k^\diamond \wedge \Sigma^\infty S^{\ell\rho})_{h\text{Aff}_k} \\
&\simeq (\mathcal{E}_k^{St} \wedge \Sigma^\infty S^{\ell\rho})_{h\Delta hGL} \\
&\simeq (GL_+ \wedge \Sigma S_{h\Delta}^{\ell\rho})_{hGL}[\varepsilon_k^{st-1}] \\
&\simeq \Sigma^\infty S_{h\Delta}^{\ell\rho} \cdot \varepsilon_k^{St} \\
&\simeq \Sigma^\ell(\Sigma^\infty S_{h\Delta}^{\ell\rho}) \text{ (piece associated with Steinberg idempotent)}
\end{aligned}$$

The Tits building is  $T_k := |\mathcal{E}'|$  w.r.t. subgroups of  $\Delta$ , the Steinberg module  $\mathcal{E}_k^{St}$  won't be defined, and  $\varepsilon_k^{St}$  is the Steinberg idempotent. The first  $\simeq$  is from last time, the second is SW duality, the third is one of the main steps.  $\square$

I won't quite prove (1), but something like it without the duality.

**Theorem 5.2.**  $P_n^\diamond \wedge S_{h\Sigma_n}^{\ell\rho} \simeq (T_k^\diamond \wedge S^{\ell\rho})_{h\text{Aff}_k}$

*Proof.* It suffices to show  $H_*^\Sigma(P_n^\diamond; \mathbb{F}_p^\pm) \cong H_*^{\text{Aff}_k}(T_k^\diamond; \mathbb{F}_p^\pm)$ , because then you can use the Thom isomorphism. Use the map  $T_k \hookrightarrow P_n$ .

Look at the diagram

$$(5.1) \quad \begin{array}{ccccc}
\Sigma \times_{\text{Aff}} |\mathcal{E}'/\Delta| & \longrightarrow & \Sigma \times_{\text{Aff}} |\mathcal{E}/\Delta| & \longrightarrow & T_k^\diamond \times_{\text{Aff}} \Sigma \\
\downarrow & & \downarrow & & \downarrow \simeq \\
|\mathcal{E}'| & \xrightarrow{q} & |\mathcal{E}| & \longrightarrow & C := \text{cofib } q \\
\downarrow & & \downarrow & & \downarrow \\
|\mathcal{P}| & \longrightarrow & * & \longrightarrow & P_n^\diamond
\end{array}$$

We want to show that  $C \rightarrow P_n^\diamond$  is an equivalence. Note that  $\mathcal{E}'/\Delta$  isn't a quotient – it's an overcategory, which you then take the nerve of (or you can think of it as a subcategory of things contained in  $\Delta$ ). The claim is that the top left square is a homotopy pushout. We want to show that the vertical maps are  $H_*^\Sigma(-; \mathbb{F}_p^\pm)$ -equivalences.

We're using Shapiro's lemma that says that “the cohomology of the transfer is the cohomology of the thing”.

So far, we've been working in  $\text{Hom}(BG^{op}, \text{Spaces})$ . We'll want to go to a better model for equivariant homotopy theory; in particular, we want to look at  $\text{Hom}(\mathcal{O}(G)^{op}, \text{Spaces})$ , where  $\mathcal{O}(G)$  is the category of transitive  $G$ -sets. This is equivalent to keeping track of the set-theoretic fixed points.

If  $\mathcal{C}$  is a collection of subgroups closed under conjugation, we can form  $O(\mathcal{C})$ , the  $G$ -sets of form  $G/K$  for  $K \in \mathcal{C}$ .

In particular,  $\text{Hom}(BG^{op}, \text{Spaces}) = \text{Psh}(\mathcal{O}(\{e\}))$ . But in general, we have

$$\text{Psh}(\mathcal{O}(\mathcal{C})) \begin{array}{c} \xrightarrow{i_!} \\ \xleftarrow{i^*} \\ \xrightarrow{i_*} \end{array} \text{Psh}(\mathcal{O}(G))$$

where the maps are given by left and right Kan extension (in an  $\infty$ -categorical sense). We'll denote  $X_{\mathcal{C}} := i_! i^* X$ , and there is a map  $X_{\mathcal{C}} \rightarrow X$ .

For example,  $X_{\{e\}} = X \times EG$  and  $*_{\{e\}} = EG$  so this is a smashing localization. (But it's not generally smashing – subconjugacy is nice, but conjugacy, which we need, is worse behaved in general.)

**Fact 5.3.** Given  $f : X \rightarrow Y$  you only need to check  $G$ -equivalence on  $\text{Iso}(X) \cup \text{Iso}(Y)$ .

Every  $X$  we care about will be as good as  $G$ -cellular, and we know  $X_{\mathcal{C}}$  commutes with things you want. So it suffices to reduce to looking at a single  $G$ -set.

$$(G/K)_{\mathcal{C}} \simeq G/K \times (*_{\mathcal{C}})_{G/K}$$

Define  $EC := *_{\mathcal{C}}$ . The fixed points are not too hard to understand: from the definition of Kan extension,

$$(EC)(K) \simeq \text{hocolim}_{((G/e)/\mathcal{O}(\mathcal{C}))^{op, K}} * \simeq |(G/\mathcal{O}(\mathcal{C}))^{op, K}|.$$

There's a functor  $(G/\mathcal{O}(\mathcal{C}))^{op} \rightarrow \mathcal{C}$  sending something to the stabilizer of an element? This is an equivalence of categories. So in particular,

$$|(G/\mathcal{O}(\mathcal{C}))^{op, K}| \simeq |K/\mathcal{C}|$$

and

$$EC(e) \simeq |\mathcal{C}|.$$

The whole reason we did all of this is to be able to plug these things back into the bottom left square in (5.1)

$$\begin{array}{ccc} |\mathcal{E}'| & \longrightarrow & |\mathcal{E}| \\ \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & * \end{array}$$

We want to understand

$$\begin{array}{ccccc} E\mathcal{E}' & \longrightarrow & E\mathcal{E} & & \\ \searrow & & \downarrow & \searrow & \\ & & |\mathcal{E}'| & \longrightarrow & |\mathcal{E}| \\ & & \downarrow & & \downarrow \\ E\mathcal{P} & \searrow & \mathcal{P} & \longrightarrow & * \\ & & \downarrow & & \downarrow \\ & & * & & * \end{array}$$

It suffices to evaluate the diagram at  $e$ , but the diagram does exist at the level of  $G$ -spaces. The missing maps are actually missing, and we need to look at another diagram.

$$\begin{array}{ccc}
E\mathcal{E}' & \longrightarrow & |\mathcal{E}'| \\
\downarrow & & \downarrow \\
E\mathcal{F}^\circ & \longrightarrow & |\mathcal{F}^\circ| \\
\uparrow & & \uparrow \downarrow \\
E\mathcal{P} & \longrightarrow & |\mathcal{P}|
\end{array}$$

Claim the bottom left map is a  $\Sigma$ -equivalence, the rightward maps are homotopy equivalences, and we want to show that the right downward composite is a homotopy equivalence. It suffices to show that  $E\mathcal{P} \rightarrow E\mathcal{F}^\circ$  and  $E\mathcal{E}' \rightarrow (E\mathcal{F}^\circ)_\Sigma$  (below) are  $\Sigma$ -equivalences.

$$\begin{array}{ccccc}
& & E\mathcal{E}' & \longrightarrow & |\mathcal{E}'| \\
& \swarrow & \downarrow & & \downarrow \\
(E\mathcal{F}^\circ)_\Sigma & \longrightarrow & E\mathcal{F}^\circ & \longrightarrow & |\mathcal{F}^\circ| \\
\downarrow & & \uparrow & & \uparrow \downarrow \\
(E\mathcal{P})_\Sigma & \longrightarrow & E\mathcal{P} & \longrightarrow & |\mathcal{P}|
\end{array}$$

This implies that the left vertical map is a  $\Sigma$ -equivalence. The last thing you need is information about  $(E\mathcal{P})_\Sigma \rightarrow E\mathcal{P}$  is a homology isomorphism. For showing  $E\mathcal{F}^\circ \rightarrow E\mathcal{P}$  a  $\Sigma$ -equivalence, it suffices for  $(E\mathcal{P})(H) \simeq *$ .  $|H \setminus \mathcal{P}| \simeq *$  for  $H \in \mathcal{F}^\circ$ . Similarly for the other map: evaluate on isotropy.

It suffices to know  $*_{\mathcal{E}} \rightarrow *$  is a  $H^G(-; \mathbb{F}_p^\pm)$ -equivalence, for  $G \subset \Sigma_n$ . This is true because the action of  $G$  on  $\mathbb{F}_p^\pm$  always contains something of order  $p$ ; then look at transfers from a  $p$ -Sylow subgroup. You want to know that  $|\mathcal{E}| \rightarrow *$  is a homology isomorphism.  $\square$

## 6. RASMUS JOHANSEN, 10/25: THE CHAIN RULE AND OPERATIONS ON LAYERS OF THE GOODWILLIE TOWER

Everything is 2-localized.

I'm going to stick to a 1-categorical framework, but you can do all of this with  $\infty$ -categories as well.

**Definition 6.1.** A symmetric sequence in  $\mathcal{C}$  is a functor  $\text{Fin}_{\neq \emptyset}^{\text{bij}} \rightarrow \mathcal{C}$ .

**Example 6.2.** The derivative of a functor  $\partial_* F$  forms a symmetric sequence.

**Definition 6.3.** Let  $\Phi$  be an operad in  $\text{Top}_*$  (a monoid in symmetric sequences). An element of  $B(\Phi)(A)$  consists of:

- (1) a rooted tree  $T$

- (2) a bijection  $\alpha : A \rightarrow \text{leaves of } T$
- (3) for every vertex  $v$ ,  $x_v \in \Phi(\text{incoming edges}_v)$
- (4) metric  $\ell$  on edges such that
  - the distance from the root to a leaf is 1
  - internal vertices have  $\geq 2$  incoming edges and exactly one outgoing edges

subject to the following equivalence relations:

- trees with  $x_v \simeq *$  for some  $v$  are  $\sim *$ ;
- if an external edge (i.e. one touching a leaf)  $e$  has  $\ell(e) = 0$ ,  $T \sim *$ ;
- if an internal edge  $e$  has  $\ell(e) = 0$ , then  $T \sim$  the tree where that edge is removed.

(E.g. draw a binary tree with the root at the top and edges pointing upwards.)

I want to think about simplices here in the general sense of  $\Delta^n \rightarrow B(\Phi)(i)$  (here  $i$  means any set of cardinality  $i$ ).

**Example 6.4.** If we have  $T \in B(\Phi)(i)$  but forget its height, and it has  $n + 2$  levels (number of distances from the root at which a vertex occurs), then we can define  $\Delta^n \rightarrow B(\Phi)(i)$ .

**Definition 6.5.**  $B(\Phi) \simeq |B_\bullet(\Phi)|$

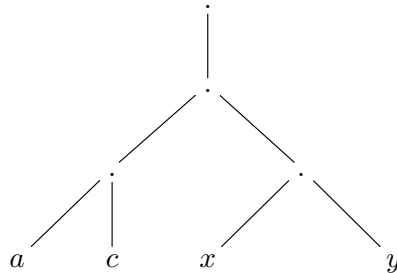
**Proposition 6.6.**  $\text{sk}_n |B_\bullet(\Phi)(i)|$  includes all trees from  $B(\Phi)(i)$  with  $\leq n + 2$  levels.

**Definition 6.7** (cooperad structure). Let  $A = \{a, b, c\}$  and  $B = \{x, y\}$ . Then a cooperad structure map looks like

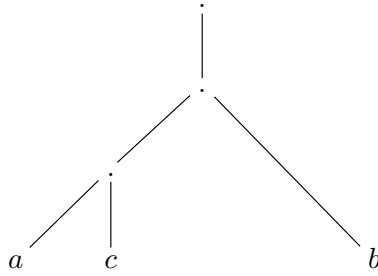
$$B(\Phi)(A \cup_b B) \rightarrow B(\Phi)(A) \wedge B(\Phi)(B)$$

where  $A \cup_b B = (A \cup B) \setminus \{b\}$ .

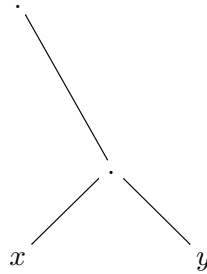
(Here we're thinking of  $B(\Phi)(A)$  as spaces.) For example,



is an element of  $B(\Phi)(A \cup_b B)$ , and this gets sent to the smash product of

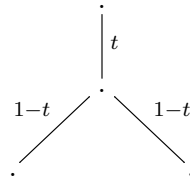


and



**Theorem 6.8.**  $\partial_i \simeq \Sigma^\infty B(\text{Comm})(i)^\vee$

**Example 6.9.** I claim that  $\partial_2 \simeq S^{-1}$ . The only nontrivial tree with 2 leaves is



Thinking about what happens when  $t \rightarrow 1$ , the space of these is  $S^1$ . So  $B(\text{Comm})(2) \simeq S^1$  (and  $\text{Comm}(i) = S^0$ ). So  $\partial_2 \simeq S^{-1}$ .

Since  $B(\text{Comm})(*)$  has a co-operad structure, its dual has an operad structure.

**Corollary 6.10.**  $\partial_*$  is an operad.

**Theorem 6.11.** If  $F$  is reduced and preserves filtered colimits, then  $\partial_* F$  is a bimodule over  $\partial_* \mathbb{1}$ . If  $G$  satisfies the same, then

$$\partial_* F \circ_{\partial_* \mathbb{1}} \partial_* G \simeq \partial_*(FG).$$

Here the product  $\circ$  is in a derived sense.

I want to construct operations

$$\overline{Q}^j : H_d(D_i(F)(X)) \rightarrow H_{d+j-1}(D_{2i}(F)(X)).$$

The way to do this is via constructing operations

$$\xi_i : \Sigma^{-1} D_i(F)(X)_{h\Sigma_2}^{\wedge 2} \simeq (\partial_2 \wedge \partial_i(F)^{\wedge 2} \wedge X^{\wedge i})_{h\Sigma_2 \wr \Sigma_i} \rightarrow D_{2i}(X).$$

We just computed above that  $\partial_2 \simeq \Sigma^{-1}$  (and corresponds to the desuspension on the LHS). We have the composition product  $\partial_* \mathbb{1} \circ \partial_* F \rightarrow \partial_* F$ . This is a bunch of terms involving partitions, and restricting to the partition where you split  $\{1, \dots, 2n\}$  as  $\{1, \dots, n\} \sqcup \{n+1, \dots, 2n\}$ , this gives rise to a map

$$\partial_2 \mathbb{1} \wedge \partial_n F \wedge \partial_n F \rightarrow \partial_{2n} F$$

which is  $\Sigma_2 \wr \Sigma_n$ -equivariant.

So our  $\xi$  operation is

$$\xi_i : \Sigma^{-1} D_i(F)(X)_{h\Sigma_2}^{\wedge 2} \simeq (\partial_2 \wedge \partial_i(F)^{\wedge 2} \wedge X^{\wedge i})_{h\Sigma_2 \wr \Sigma_i} \rightarrow (\partial_{2i} F \wedge X^{\wedge 2i})_{h\Sigma_{2i}} \simeq D_{2i}(X).$$

Now to define  $\overline{Q}^i$ 's, draw the diagram

$$\begin{array}{ccc} H_d(D_i(F)(X)) & \xrightarrow{\overline{Q}^j} & H_{d+j-1}(D_{2i}(F)(X)) \\ \downarrow Q^j & & \uparrow (\xi_i)_* \\ H_{d+j}(D_i(F)(X)_{h\Sigma_i}^{\wedge 2}) & \xrightarrow[\simeq]{\sigma^{-1}} & H_{d+j-1}(\Sigma^{-1} D_i(F)(X)_{h\Sigma_2}^{\wedge 2}) \end{array}$$

Here  $\sigma^{-1}$  is a desuspension and the dotted arrow is defined by going around the other arrows.

**Definition 6.12.** Define  $\overline{R}_n$  to be the free  $\mathbb{F}_2$ -algebra generated by  $\overline{Q}^j$  for  $j \geq 0$  modulo

- (1)  $\overline{Q}^{j_1} \overline{Q}^{j_2} \dots \overline{Q}^{j_k} = 0$  for  $j_1 < j_2 + \dots + j_k + n$
- (2)  $\overline{Q}^r \overline{Q}^s = \sum_{\ell=0}^{r-s-1} \binom{2s-r+1+2\ell}{\ell} \overline{Q}^{2s+1+\ell} \overline{Q}^{r-s-1-\ell}$  if  $s < r < 2s$ .

The proof has something to do with chasing  $\overline{Q}^{j_1} \overline{Q}^{j_2} \dots \overline{Q}^{j_k}$  through the following diagram:

$$\begin{array}{c} \Sigma^{-1} D_{2^k}(S^n)_{h\Sigma_2}^{\wedge 2} \\ \downarrow \\ \Sigma^{-1} \text{Map}_*(\text{sk}_k |B(\Phi)(2^k)|, S^{n^{2^k}})_{h\Sigma_2 \wr \Sigma_{2^k}}^{\wedge 2} \\ \uparrow \\ \Sigma^{-1} (\Sigma^{-k} (S^n)_{h\Sigma_2^{i^k}}^{\wedge 2^k})_{h\Sigma_2}^{\wedge 2} \end{array}$$

**Theorem 6.13.** As a module over  $\overline{R}_n$  we have

$$\bigoplus_{k \geq 0} H_*(D_{2^k}(S^n)) = \overline{R}_n \{\iota_n\}$$

where  $\iota_n \in H_n(D_1 S^n) = \widetilde{H}_n(S^n)$  is the fundamental class and operations behave as in  $\overline{R}_n$ .

The idea is that we're doing a similar thing as Arone-Mahowald, but we need the operad stuff to get the algebra structure as opposed to Arone-Mahowald which was just getting the module structure.

This is based on Behrens' memoir, sections 2.1-2.4 and part of A.1-A.3.

There are three main ingredients:

- (1) The EHP sequence comes from the sequence of functors  $\mathbb{1} \rightarrow \Omega\Sigma \rightarrow \Omega\Sigma Sq$  which, when applied to a 2-local sphere, gives a fiber sequence and hence a spectral sequence

$$E_{m,t}^1 \cong \pi_{t+m+1}(S^{2m+1}) \implies \pi_t^s.$$

Truncating at  $m \geq n$ , you get a spectral sequence  $E_{m,t}^1 \implies \pi_{t+n}(S^n)$  computing unstable homotopy.

- (2) There is a fiber sequence

$$\Sigma^n L(k-1)_{2n+1} \rightarrow L(k)_n \rightarrow L(k)_{n+1}$$

where the  $L(k)$ 's came from  $\mathbb{D}_{2^k}(S^n) = \Sigma^{\pm(n-k)} L(k)_n$ . We'll construct an Atiyah-Hirzebruch spectral sequence for the  $L(k)_n$ 's.

- (3) The Goodwillie spectral sequence lets us compute  $\pi_*(S^n)$  in terms of  $\pi_*(L(k)_n)$ .

Outline:

- (1) fiber sequence of  $L(k)_n$ 's
- (2) homology of  $L(k)_n$ 's
- (3) sequences of spectral sequences ("transfinite" AHSS and GSS)

**7.1. Fiber sequences of  $L(k)_n$ 's.** We have a sequence of functors  $\mathbb{1} \rightarrow \Omega\Sigma \rightarrow \Omega\Sigma Sq$ , which when applied to the 2-local sphere, gives a fiber sequence. I want to say that  $P_i$ (this) also gives a fiber sequence when applied to the 2-local sphere.

**Proposition 7.1.** *There are fiber sequences*

$$\begin{aligned} P_i(\mathbb{1})S^n &\rightarrow P_i(\Omega\Sigma)(S^n) \rightarrow P_i(\Omega\Sigma Sq) \\ \mathbb{D}_i(\mathbb{1})S^n &\rightarrow \mathbb{D}_i(\Omega\Sigma)(S^n) \rightarrow \mathbb{D}_i(\Omega\Sigma Sq) \end{aligned}$$

The proof uses orthogonal calculus (Weiss calculus). It's not a fiber sequence of functors (i.e. it's not a fiber sequence when applied to any space) so you can't get this from general results, but it's a fiber sequence on things you care about, like spheres.

*Proof.* Let  $F : \text{Top}_* \rightarrow \text{Top}_*$  be a topologically enriched functor. Then consider the functor  $\widetilde{F}_X : \text{Vect} \rightarrow \text{Top}_*$  given by  $V \mapsto \Omega^V F \Sigma^V(X)$ . Weiss proves that

$$P_i^{\text{orth}}(\widetilde{F}_X)(V) \simeq \Omega^V P_{i+1} \Sigma^V X$$

and the same for  $\mathbb{D}_i$ . If  $X = S^n$ , then  $\widetilde{\mathbb{1}} \rightarrow \widetilde{\Omega\Sigma} \rightarrow \widetilde{\Omega\Sigma Sq}$  is a fiber sequence of functors  $\text{Vect} \rightarrow \text{Top}_*$ . General facts about Weiss calculus then gives that you get a fiber sequence on  $P_i$  etc. Now take  $V$  to be the zero vector space.  $\square$

**Lemma 7.2.** *Let  $F : \text{Top}_* \rightarrow \text{Top}_*$  be a reduced finitary homotopy functor, stably  $i$ -excisive for all  $i$ . Then*

$$P_i(FSq) \simeq P_{\lfloor \frac{i}{2} \rfloor}(F)Sq$$

$$\mathbb{D}_i(FSq) \cong \begin{cases} \mathbb{D}_{\frac{i}{2}}(F)Sq & i \text{ even} \\ * & \text{else.} \end{cases}$$

*Proof.* The second statement follows from the first because of how the  $\mathbb{D}_i$ 's are defined. It suffices to show:

- (1)  $P_{\lfloor \frac{i}{2} \rfloor}(F)Sq$  is  $i$ -excisive.
- (2)  $F Sq$  agrees with  $P_{\lfloor \frac{i}{2} \rfloor}(F)Sq$  to order  $i$ .

Recall  $F, G$  agree to order  $i$  if there exists a map  $F \rightarrow G$  and constants  $c, K$  such that for all  $X$ ,  $\text{conn } X \geq K$  and  $FX \rightarrow GX$  is  $(-c + (i+1)\text{conn}(X))$ -connected. (The map is highly connected for sufficiently highly connected things.)

If  $F, G$  are stably  $i$ -excisive for all  $i$ , then:  $F$  and  $G$  agree to order  $n$  iff  $P_n F \simeq P_n G$ .

(1) is straightforward, so we'll do (2).  $F$  and  $P_{\lfloor \frac{i}{2} \rfloor} F$  agree to order  $\lfloor \frac{i}{2} \rfloor$ , so there exists  $c$  such that for  $Y$  highly connected,  $FY \rightarrow P_{\lfloor \frac{i}{2} \rfloor}(F)(Y)$  is  $(-c + (\lfloor \frac{i}{2} \rfloor + 1)\text{conn } Y)$ -connected. If  $Y = X \wedge X$  and  $X$  is highly connected, then  $FX \wedge X \rightarrow P_{\lfloor \frac{i}{2} \rfloor}(F)(X \wedge X)$  is  $(-c + 2(\lfloor \frac{i}{2} \rfloor + 1)\text{conn } X)$ -connected, and so it's  $(-c + (i+1)\text{conn } X)$ -connected.  $\square$

**Corollary 7.3.** *The fiber sequences of the previous proposition are equivalent to*

$$P_{2m}(S^n) \xrightarrow{E} \Omega P_{2m}(S^{n+1}) \xrightarrow{H} \Omega P_m(S^{2n+1})$$

$$P_{2m+1}(S^n) \xrightarrow{E} \Omega P_{2m+1}(S^{n+1}) \xrightarrow{H} \Omega P_m(S^{2n+1})$$

and similarly for  $\mathbb{D}_i$ .

*Proof.*

$$P_i(\Omega F) \xrightarrow{w} \Omega P_i(F)$$

$$P_i(F\Sigma) \xrightarrow{\text{susp}} P_i(F)(\Sigma)$$

$$\mathbb{D}_i(\Omega F) \simeq \Sigma^{-1} \mathbb{D}_i(F)$$

$\square$

**Corollary 7.4.** *There are fiber sequences*

$$(7.1) \quad \Sigma^n L(k-1) \xrightarrow{P} L(k)_n \xrightarrow{E} L(k)_{n+1}$$

*Proof.* I will show  $\mathbb{D}_{2k}(S^n) = \Sigma^{n-k} L(k)_n$ . To do this, apply  $\Sigma^{-(n-k)+1}$  to the  $\mathbb{D}_i$  version of the fiber sequence in the corollary.  $\square$



**7.2. Homology of  $L(k)_n$ 's.** For a sequence  $J = (j_1, \dots, j_k)$  define

$$[j_1, \dots, j_k] := \sigma^{k-n} \overline{Q}^{j_1} \dots \overline{Q}^{j_k} \iota_n \in H_*(L(k)_n).$$

Previously we proved:

$H_*(L(K)_n) \cong \mathbb{F}_2\{[j_1, \dots, j_k] : J = (j_1, \dots, j_k) \text{ is completely inadmissible, } |J| = K, e(J) \leq n\}$   
where  $e$  denotes excess.

**Proposition 7.5.**

$$P_*([j_1, \dots, j_{k-1}]) = [j_1, \dots, j_{k-1}, n]$$

$$E_*([j_1, \dots, j_k]) = \begin{cases} 0 & \text{if } j_k = n \\ [j_1, \dots, j_k] & \text{otherwise.} \end{cases}$$

and the fiber sequence (7.1) induces a SES on homology.

The key lemma is as follows:

**Lemma 7.6.**

$$\begin{aligned} \text{susp}_*(\overline{Q}^j x) &= (\overline{Q}^j \text{susp}_*)(x) \\ w_*(\overline{Q}^j x) &= (\overline{Q}^j w_*)(x) \\ \text{sqrt}_*(\overline{Q}^j x) &= (\overline{Q}^j \text{sqrt}_*)(x) \end{aligned}$$

where  $\text{sqrt} : \mathbb{D}_{2n}(FSq) \rightarrow \mathbb{D}_n(F)(Sq)$ .

*Proof.*  $\text{susp}_*$  gives  $\partial_*(FG) \simeq B(\partial_*F, \partial_*\mathbf{1}, \partial_*G)$  where  $S^*$  is the symmetric sequence that assigns  $n \mapsto S^n$ . Using the chain rule,

$$\partial_*(F\Sigma) \simeq B(\partial_*F, \partial_*\mathbf{1}, \partial_*\Sigma) \simeq \partial_*F \wedge S^*$$

So you get a commutative diagram

$$\begin{array}{ccc} \partial_2(\mathbf{1}) \wedge_{h\Sigma_2} \mathbb{D}_i(F\Sigma)(X)^{\wedge 2} & \longrightarrow & \mathbb{D}_{2i}(F\Sigma)(X) \\ \simeq \downarrow & & \downarrow \simeq \\ \partial_2(\mathbf{1}) \wedge_{h\Sigma_2} \mathbb{D}_i(F)(\Sigma X)^{\wedge 2} & \longrightarrow & \mathbb{D}_{2i}(F)(\Sigma X) \end{array}$$

□

Next, we'll construct an Atiyah-Hirzebruch-type spectral sequence for the  $L(K)_n$ 's. We define

$$L(K)_n^m := \text{fib}(L(K)_n \xrightarrow{E^{m-n+1}} L(K)_{m+1}).$$

For example,  $L(1)_n^m$  are truncated projective spaces  $P_n^m$ .

**Proposition 7.7.**

$$H_*(L(K)_n^m) = \mathbb{F}_2\{[J] : J \text{ c.u.}, |J| = K, n \leq e(J) \leq m\}$$

There is an equivalence

$$\operatorname{colim}(\cdots \rightarrow L(K)_n^m \rightarrow L(K)_n^{m+1} \rightarrow L(K)_n^{m+2} \rightarrow \cdots) \xrightarrow{\cong} L(K)_n$$

where the filtration quotients are given by

$$L(K)_n^{m-1} \xrightarrow{E^p} L(K)_n^m \rightarrow \Sigma^m L(k-1)_{2m+1}.$$

This gives rise to a spectral sequence

$$E_{*,t}^1 = \bigoplus_{m \geq n} \pi_t(\Sigma^m L(k-1)_{2m+1}) \implies \pi_t(L(K)_n).$$

If you iterate this, you get a sequence of spectral sequences for the different  $K$ 's, and if you put them all together you get

$$\bigoplus_{\substack{(j_1, \dots, j_k) \text{ c.u.} \\ j_k \geq n}} \pi_t(S^{j_1 + \cdots + j_k}) \implies \bigoplus_{\substack{(j_2, \dots, j_k) \text{ c.u.} \\ j_k \geq n}} \pi_t(\Sigma^{j_2 + \cdots + j_k} L(1)_{2j_2+1}) \implies \cdots \implies \pi_t(L(K)_n).$$

This is kind of unwieldy; Behrens deals with this using “transfinite spectral sequences.” The idea is that there are two directions, one of which is indexed by  $\mathbb{N}$  and the other is larger:

$$E_{t,\beta}^\alpha$$

where  $\alpha, \beta$  are in a Grothendieck group of ordinals and  $t \in \mathbb{N}$ . (You might only need the full generality for the Goodwillie spectral sequence.)

The transfinite AHSS (TAHSS) is:

$$(7.2) \quad E_{t,\mu}^1(L(K)_n) = \begin{cases} \pi_t(S^{\|J\|}) & \mu \text{ corresponds to a c.u. sequence} \\ 0 & \text{else.} \end{cases} \implies \pi_t(L(K)_n)$$

(By “corresponds to”, I mean  $\mu = j_0 + j_1\omega + \cdots + j_k\omega^{k-1}$  and  $(j_1, \dots, j_k)$  is completely unadmissible.) (Idea: this is a refinement of the AHSS where instead of building up  $L(K)$  by cells (spheres), you build it up by  $L(K-1)$ -cells.)

The transfinite Goodwillie spectral sequence (TGSS)

$$\bigoplus_{k \geq 0} \bigoplus_{\substack{(j_1, \dots, j_k) \text{ c.u.} \\ k \geq n}} \pi_t(S^{j_1 + \cdots + j_k}) \implies \cdots \implies \bigoplus_{k \geq 0} \pi_t(L(K)_n) \implies \pi_{t+n-k}(S^m)$$

which corresponds to

$$E_{t,\mu}^1(S^n) = \begin{cases} \pi_t(S^{n-|J|+\|J\|}) & \mu = \mu(J), J \text{ c.u., } |J| \geq 0, e(J) \geq n \\ 0 & \text{else} \end{cases} \implies \pi_*(S^n).$$