LECTURE 3: EXAMPLES OF GOODWILLIE CALCULUS

In these notes, we'll (subconsciously) switch between saying "∞-category" and "category". Feel free to generously pour — or rip out, whatever you fancy — the prefix "∞-" throughout.

1. Generalities

Last time, we stated the following result.

Theorem 1. *Let $\mathcal{C}$ be an ∞-category with finite colimits and $\mathcal{D}$ be a differentiable ∞-category (i.e., it admits finite limits and colimits of diagrams of the form $(\mathbf{Z}, \geq)$, such that these commute). Then the inclusion*

$$\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

has a left adjoint, denoted P_n. Moreover, P_n preserves finite limits.

There are some straightforward (formal) consequences of this theorem, that we'll discuss below. One immediate result is that we get a natural transformation $P_n \to P_{n-1}$; moreover, it follows that

$$P_n P_{n+k} F \simeq P_n F$$

for any $k \geq n$. Before we get to more exciting applications, we need to find a way to study polynomial approximations to functors.

When working with actual smooth functions, the nth Taylor approximation (around 0) to $f : \mathbf{R} \to \mathbf{R}$ is given by

$$p_n(x) = \sum_{i=0}^{n} f^{(n)}(0) \frac{x^n}{n!}.$$

In particular, the difference between two consecutive Taylor approximations is given by

$$p_n(x) - p_{n-1}(x) = f^{(n)}(0) \frac{x^n}{n!}.$$

The analogue of taking the "difference", when working with (stable) ∞-categories, is to take the fiber of the map $P_n \to P_{n-1}$.

Definition 2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, with the same assumptions on $\mathcal{C}$ and $\mathcal{D}$ as above. Define $D_n F$ to be the fiber[1] of the natural transformation $P_n F \to P_{n-1} F$.

Proposition 3. *$D_n F$ is homogeneous of degree n, i.e., it is n-excisive, and*

$$P_{n-1} D_n F(X) \simeq *$$

for all $X \in \mathcal{C}$.

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[1] This exists since fiber sequences are just finite limits, which we assumed exist in $\mathcal{D}$.

Proof. We know that $P_{n-1}F$ is n-excisive, and so, since P_n preserves finite limits, it must send the fiber sequence defining D_nF to the fiber sequence

$$P_nD_nF \to P_nP_nF \simeq P_nF \to P_nP_{n-1}F \simeq P_{n-1}F,$$

which evidently implies that $P_nD_nF \simeq D_nF$. We get the second statement by a similar argument: apply P_{n-1} to the fiber sequence defining D_nF; since $P_{n-1}P_nF \simeq P_{n-1}F$, we have a fiber sequence

$$P_{n-1}D_nF \to P_{n-1}P_nF \simeq P_{n-1}F \to P_{n-1}P_{n-1}F \simeq P_{n-1}F,$$

which implies that $P_{n-1}D_nF(X) \simeq *$ for all $X \in \mathcal{C}$. □

Having constructed a tower of fibrations, we, as homotopy theorists, are naturally inclined to apply homotopy to obtain an exact couple, and hence a spectral sequence. The resulting Bousfield-Kan spectral sequence is of signature

$$E^1_{p,q} = \pi_p D_q F(X) \Rightarrow \pi_{p+q} P_\infty F(X). \tag{1}$$

Note that we don't necessarily have strong convergence; if the map $F \to P_\infty F$ is an equivalence, we get conditional convergence. If the spectral sequence had a vanishing line of positive slope, we would get strong convergence. In particular, the following (technical) result[2] gives sufficient conditions for the existence of such a vanishing line.

Lemma 4. *If F is a "n-analytic functor" — this is like saying what the radius of convergence is, i.e., that F "agrees" P_jF for sufficiently large j — and X is k-connected for some $k > n$, then the map $F(X) \to P_qF(X)$ is at least $(d + k + q(k - n))$-connected for some d.*

It follows that $D_qF(X)$ is $(d + k + (q - 1)(k - n))$-connected, and hence $E^1_{p,q}$ vanishes for

$$q \geq (p - d - k)/(k - n) + 1$$
$$p \leq d + k + (q - 1)(k - n).$$

Overall, we get a vanishing line with positive slope, since we can rewrite the inequality for q as

$$q \geq \frac{1}{k-n}p - \frac{d+k}{k-n}.$$

This was why we assumed that $k > n$: otherwise, the vanishing line wouldn't have positive slope.

When we worked with the homogeneous component in $p_n(x)$ of degree n, we had a term of the form

$$\text{derivative} \cdot \frac{x^n}{n!}. \tag{2}$$

A similar description is true in Goodwillie calculus.

Theorem 5. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor as above which is homogeneous of degree n, whose target is a* stable *∞-category, then there is a symmetric monoidal n-linear*[3] *functor $LF : \mathcal{C}^n \to \mathcal{D}$, and a weak equivalence*

$$LF(X, \cdots, X)_{h\Sigma_n} \simeq F(X).$$

This will allow us to write down a formula for $D_nF(X)$, which will look very similar to the degree n component of the Taylor polynomial. We first need a categorical result.

[2] See Theorem 10.1.51 of Munson-Volic's *Cubical homotopy theory.*

[3] This means that it is homogeneous of degree 1 in each variable, and is invariant under permutations of the coordinates.

Lemma 6. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories as above, tensored over* Top_*. *If $F : \mathcal{C} \to \mathcal{D}$ is a continuous functor, there is a natural* assembly map

$$F(X) \otimes K \to F(X \otimes K),$$

where $X \in \mathcal{C}$ and $K \in \mathrm{Top}_*$, which is the identity if $K = S^0$.

Proof. If F is continuous, the function

$$F : \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

is continuous. This allows us to push forward the identity $X \otimes K \to X \otimes K$ via the following adjunctions:

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(X \otimes K, X \otimes K) &\simeq \mathrm{Map}_{\mathcal{C}}(K, \mathrm{Map}(X, X \otimes K)) \\ &\xrightarrow{F_*} \mathrm{Map}_{\mathcal{D}}(K, \mathrm{Map}(F(X), F(X \otimes K))) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(K \otimes F(X), F(X \otimes K)).\end{aligned}$$

Continuity was necessary, since the map $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$ needed to be pointed, and continuity gets us that $X \to * \to Y$ is sent to $F(X) \to * \to F(Y)$. □

This assembly map, under the identification in Theorem 5, can be collected together to get a map

$$(LF(X, \cdots, X) \otimes K^{\wedge n})_{h\Sigma_n} \to LF(X \otimes K, \cdots, X \otimes K)_{h\Sigma_n}.$$

Now, suppose $\mathcal{C} = \mathrm{Top}_*$ and $\mathcal{D} = \mathrm{Sp}$, and let $X = S^0$; then, if we let

$$C_F(n) = LF(S, \cdots, S),$$

we get a map

$$(C_F(n) \wedge K^{\wedge n})_{h\Sigma_n} \to LF(K, \cdots, K)_{h\Sigma_n}.$$

If F is D_nG, we'll write $C_G(n)$ instead of $C_F(n)$; the identification in Theorem 5 now gives a map

$$(C_F(n) \wedge K^{\wedge n})_{h\Sigma_n} \to D_nF(K).$$

If $K = S^0$, this is the identity — it was constructed that way — and so this map is an equivalence for all finite K. Alternatively, if F preserved all filtered homotopy colimits, this map would be an equivalence for all K. This gives us a nice analogue of Equation (2):

Corollary 7. *If K is a finite complex, or if F commutes with all filtered homotopy colimits, there is a homotopy equivalence*

$$(C_F(n) \wedge K^{\wedge n})_{h\Sigma_n} \to D_nF(K).$$

We'll see some examples of this below.

2. Another take on spectra

Goodwillie calculus can be used to provide another, somewhat amusing — but helpful — description of the category of spectra.

Proposition 8. *Let $\mathcal{C}$ be a pointed ∞-category with finite limits. Then there is an equivalence*

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}^1(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{C}).$$

Proof sketch. Let $F : \mathrm{Top}_*^{\mathrm{fin}} \to \mathcal{C}$ be a 1-excisive functor; then $\{F(S^n)\}_{n \geq 0}$ is a spectrum object in $\mathcal{C}$. □

Returning back to our original situation, let $\mathcal{C}$ be an ∞-category with finite colimits and $\mathcal{D}$ be a differentiable ∞-category. Given a functor $F : \mathcal{C} \to \mathrm{Sp}(\mathcal{D})$, we can construct the functor

$$\Omega^\infty F : \mathcal{C} \to \mathrm{Sp}(\mathcal{D}) \xrightarrow{\Omega^\infty} \mathcal{D}.$$

Under the identification in Proposition 8, the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ sits inside a commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(\mathcal{D}) \arrow[rr, "\sim"] \arrow[dr, "\Omega^\infty"'] & & \mathrm{Exc}^1(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{D}) \arrow[dl, "\mathrm{ev}_{S^0}"] \\ & \mathcal{D} & \end{tikzcd}$$

It follows that

$$\Omega^\infty F = \mathrm{ev}_{S^0} \circ F.$$

For any space $X \in \mathrm{Top}_*^{\mathrm{fin}}$, we have the functor

$$\mathrm{ev}_X : \mathrm{Exc}^1(\mathrm{Top}_*^{\mathrm{fin}}, \mathcal{D}) \to \mathcal{D}.$$

We also have an identification[4]

$$P_n(\mathrm{ev}_X \circ F) = \mathrm{ev}_X \circ P_n F,$$

since $P_n F$ can be constructed explicitly in terms of filtered colimits (and ev_X commutes with these); this means that F is n-excisive (resp. homogeneous of degree n) if and only if $\mathrm{ev}_X \circ F$ is n-excisive (resp. homogeneous of degree n) for all $X \in \mathcal{C}$.

The discussion above gives equivalences

$$\mathrm{Exc}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})),\ \mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})).$$

It turns out that the functor $\Omega : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ is an equivalence when $\mathcal{D}$ is pointed, so that $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ is a stable ∞-category. This means that

$$\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}),$$

and that this functor is given by composing with $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$. This allows us to (essentially) remove the stability assumption in Theorem 5.

Any functor $F : \mathcal{C} \to \mathcal{D}$ naturally produces a homogeneous functor of degree n: take the homogeneous component $D_n F$ of degree n. The discussion above begets a lift in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, dashed] & \mathcal{D} \\ & \mathrm{Sp}(\mathcal{D}) \arrow[u, "\Omega^\infty"'] \end{tikzcd}$$

In particular, this tells us that there is a functor $\widetilde{D_n F} : \mathcal{C} \to \mathrm{Sp}(\mathcal{D})$ such that

$$D_n F(X) \simeq \Omega^\infty \widetilde{D_n F}(X).$$

It's easy to see that $\widetilde{D_n F}$ is also homogeneous of degree n.

Maybe it's time to talk about examples.

[4]More generally, if G is a functor $\mathcal{D} \to \mathcal{D}'$ between differentiable ∞-categories which preserves finite limits and sequential colimits, there is a canonical equivalence

$$P_n(G \circ F) = G \circ P_n(F).$$

3. The Snaith splitting

Let X be a topological space; then the stable homotopy groups of X can be identified with the homotopy groups of a certain space constructed from X:

$$\pi_*^s X \simeq \pi_* \Omega^\infty \Sigma^\infty X.$$

The spaces $\Omega^n \Sigma^n X$ are hard to play with. When $n = 1$, the James construction gives a description of this space: take the disjoint union $\coprod_{n \geq 0} X^n$, and quotient out by the equivalence relation that makes the basepoint the identity.

In general, May (in his work on the delooping story) showed that

$$\Omega^n \Sigma^n X \simeq \left(\coprod_{i \geq 0} C^{(n)}(i) \times_{\Sigma_i} X^i \right) / \sim,$$

where $\sim$ is the equivalence relation that makes the basepoint the identity, and $C^{(n)}(i)$ is the space of embeddings of i little n-cubes inside a big n-cube (such that they have disjoint interiors). We'll write this as $C_n(X)$; truncating the disjoint union at level k gives a subspace $F_k C_n(X)$, and there's a filtration of $C_n(X)$ coming this way. The goal of this section is prove (most of) the following splitting theorem.

Theorem 9 (Snaith)**.** *There is a splitting*

$$\Sigma^\infty \Omega^n \Sigma^n X = \bigvee_{k \geq 1} \Sigma^\infty \left(C^{(n)}(k)_+ \wedge X^{\wedge k} \right)_{h\Sigma_k}.$$

When n goes to ∞, the space $C^{(n)}(k)$ models $E\Sigma_k$, which is weakly equivalent to a point, so we get an identification

$$\Sigma^\infty \Omega^\infty \Sigma^\infty X = \bigvee_{k \geq 1} \Sigma^\infty \left(S^0 \wedge X^{\wedge k} \right)_{h\Sigma_k}.$$

Since S^0 is the unit for the smash product on pointed spaces, we get the "usual" Snaith splitting.

Let's get to work, then. Fix a finite CW-complex K. Our functor $F : \mathrm{Top}_* \to \mathrm{Sp}$ will be the functor sending

$$X \mapsto \Sigma^\infty \mathrm{Top}_*(K, -).$$

Theorem 10 (Goodwillie)**.** *This functor is analytic; in other words, the Goodwillie tower for F converges to F.*

Let $\mathcal{E}$ be the category of finite sets $\mathbf{n} = \{1, \cdots, n\}$ with $n \geq 1$ and surjections, and let $\mathcal{E}_d$ denote the full subcategory of objects of $\mathcal{E}$ of cardinality at most d. Given a topological space X, we can define a functor $X^\wedge : \mathcal{E}^{op} \to \mathrm{Top}_*$ by

$$\mathbf{n} \mapsto X^{\wedge n}.$$

This gives a functor $\mathcal{E}_d^{op} \to \mathrm{Top}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}$, also denoted $X^\wedge$.

Theorem 11 (Arone)**.** *There is an identification*[5]

$$P_d F(X) = \mathrm{Map}_{\mathrm{Fun}(\mathcal{E}_d^{op}, \mathrm{Sp})}(\Sigma^\infty K^\wedge, \Sigma^\infty X^\wedge).$$

[5]Note that this is actually a spectrum, since the ∞-category $\mathrm{Fun}(\mathcal{E}_d^{op}, \mathrm{Sp})$ is enriched over Sp.

There's always a natural transformation $F \to P_d F$; in our case, the map $F(X) \to P_d F(X)$ is given by

$$(f : K \to X) \mapsto (\Sigma^\infty \circ f^\wedge : \Sigma^\infty \circ K^\wedge \to \Sigma^\infty \circ X^\wedge).$$

Recall that $D_d F$ sits inside a tower of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
D_d F(X) \arrow[r] & P_d F(X) \arrow[d] \\
D_{d-1} F(X) \arrow[r] & P_{d-1} F(X) \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_0 F(X).
\end{tikzcd}$$

It turns out that if $K = S^n$ and $X = \Sigma^n Y$ for some space Y, this tower strongly splits, so Theorem 10 implies that

$$F(\Sigma^n Y) = \bigvee_{k \geq 0} D_d F(\Sigma^n Y).$$

When $K = S^n$, though, the functor F is simply given by

$$F(X) = \Sigma^\infty \mathrm{Top}_*(S^n, \Sigma^n Y) = \Sigma^\infty \Omega^n \Sigma^n Y.$$

Theorem 9 would follow if we could prove that $D_d F(X)$ was of the form indicated.

By Theorem 11, there's a homotopy pullback in spectra:

$$\begin{tikzcd}
P_d F(X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty K^{\wedge d}, \Sigma^\infty X^{\wedge d})_{h\Sigma_d} \arrow[d] \\
P_{d-1} F(X) \arrow[r] & \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty \delta_d K, \Sigma^\infty X^{\wedge d})_{h\Sigma_d},
\end{tikzcd}$$

where $\delta_d(K)$ is the "fat diagonal":

$$\delta_d(K) = \{(x_1, \cdots, x_d) \in K^d | \exists i \neq j \text{ such that } x_i = x_j\}.$$

If we define

$$K^{(d)} = K^{\wedge d} / \delta_d(K),$$

then the homotopy fiber $D_d F(X)$ of the map $P_d F(X) \to P_{d-1} F(X)$ is given, by the fiber of the map

$$\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty K^{\wedge d}, \Sigma^\infty X^{\wedge d})_{h\Sigma_d} \to \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty \delta_d K, \Sigma^\infty X^{\wedge d})_{h\Sigma_d},$$

since the homotopy fibers of two parallel maps in a homotopy pullback square are equivalent. This tells us that there's an equivalence

$$D_d F(X) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty K^{(d)}, \Sigma^\infty X^{\wedge d})_{h\Sigma_d}.$$

We can take the equivariant Spanier-Whitehead dual to get an equivalence

$$D_d F(X) \simeq \left(\mathbf{D}\Sigma^\infty K^{(d)} \wedge \Sigma^\infty X^{\wedge d}\right)_{h\Sigma_d}.$$

Thus, in the notation of Corollary 7, we can identify

$$C_F(d) = \mathbf{D}\Sigma^\infty K^{(d)}.$$

Before we conclude the proof of Theorem 9, let us briefly talk about the spectral sequence of Equation (1). We didn't define what an analytic functor is, but to study the inequalities in Lemma 4, we only need the following theorem of Goodwillie's.

Theorem 12. *The functor F is n-analytic, where $n = \dim K$.*

Lemma 4, and the following discussion, implies that, if X is an k-connected space for $k > n$, the space $D_q F(X)$ is $(d + k + (q-1)(k-n))$-connected. But

$$d + k + (q-1)(k-n) = d + qk - qn + n.$$

Proposition 13. *In fact, $d = q - 1 - n$. In other words, $D_q F(X)$ is N-connected, where*

$$N = (q-1-n) + qk - qn + n = (1 + k - n)q - 1.$$

Proof. Since X is k-connected, its bottom cell is in dimension $n+1$, so the bottom cell of $X^{\wedge q}$ is in dimension $q(k+1)$. Since $\dim K = n$, it follows that $\dim K^{(q)} = qn$, i.e., $\mathbf{D}K^{(q)}$ has bottom cell in dimension $-qn$. This means that $\mathbf{D}K^{(q)} \wedge X^{\wedge q}$, and hence $D_q F(X) = (\mathbf{D}K^{(q)} \wedge X^{\wedge q})_{h\Sigma_q}$, has bottom cell in dimension $(1+k-n)q$, so it is $(1+k-n)q-1$-connected. □

This immediately gives us information about the strong convergence of the spectral sequence of Equation (1).

Specializing to the case $K = S^n$, we now have to determine $\mathbf{D}\Sigma^\infty S^{n(d)}$. Let $c \in C^{(n)}(d)$; this can be viewed an inclusion

$$\coprod_{i=1}^{d} I^n \hookrightarrow I^n;$$

the Pontryagin-Thom collapse gives a map

$$S^n \to \bigvee_{i=1}^{d} S^n.$$

Overall, this procedure begets

$$\alpha(n,d) : C^{(n)}(d) \to \mathrm{Map}_{\mathrm{Top}_*}\left(S^n, \bigvee_{i=1}^{d} S^n\right).$$

Let $\delta(n,1) : C^{(n)}(1) \wedge S^n \to S^n$ be the adjoint to $\alpha(n,1)$. We have a natural inclusion $C^{(n)}(d) \hookrightarrow C^{(n)}(1)^d$, and the composite

$$\delta(n,1)^{\wedge d} : C^{(n)}(d)_+ \wedge S^{nd} \hookrightarrow C^{(n)}(1)_+^{\wedge d} \wedge S^{nd} \to S^{nd}$$

sends the fat diagonal $C^{(n)}(d)_+ \wedge \delta_d(S^n)$ to the basepoint. All in all, we've constructed a (Σ_d-equivariant) map

$$\delta(n,k) : C^{(n)}(d)_+ \wedge S^{n(d)} \to S^{nd}.$$

Theorem 14 (Ahearn-Kuhn)**.** *The map $\delta(n,k)$ induces a Σ_d-equivariant equivalence*

$$\Sigma_+^\infty C^{(n)}(d) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty S^{n(d)}, S^{nd}).$$

The proof is rather technical, so I'll omit it. But Theorem 14 is exactly what we desired; it says that

$$\mathbf{D}\Sigma^\infty S^{n(d)} = \Sigma^{nd}\Sigma^\infty_+ C^{(n)}(d),$$

which immediately implies the identification we wanted:

$$\begin{aligned} D_d F(\Sigma^{-n}X) &\simeq \Sigma^\infty\left(\mathbf{D}\Sigma^\infty S^{n(d)} \wedge \Sigma^\infty(\Sigma^{-n}X)^{\wedge d}\right)_{h\Sigma_d} \\ &\simeq \Sigma^\infty\left(C^{(n)}(d)_+ \wedge X^{\wedge d}\right)_{h\Sigma_d}. \end{aligned}$$

It's amusing to observe that when $n \to \infty$, the functor F (associated to $K = S^n$) has its homogeneous component of degree d given by the dth symmetric power — both denoted by the same symbol! It seems unlikely that Goodwillie knew that you could get the Snaith splitting from his work, so this is a happy coincidence.

4. Almost the Kahn-Priddy theorem

The functor F on spectra sending X to $\Sigma^\infty\Omega^\infty X$ is like taking the group ring. The Goodwillie tower of this functor can be studied using the example studied above. If X_n denotes the nth space of X, then

$$\Omega^n X_n \simeq \Omega^\infty X.$$

In addition, the map

$$\operatorname{colim}_n \Sigma^{-n}\Sigma^\infty X_n \to X$$

is an equivalence. The above analysis (see Theorem 12 and Lemma 4) implies that for 0-connective spectra, the Goodwillie tower of F converges. Moreover, we can identify

$$\operatorname{colim}_n \Sigma^{-n} F_d^{S^n}(X_n) \simeq F(X),$$

where F^{S^n} denotes the functor F (in the previous section) associated to $K = S^n$. Arguing as in the discussion after Theorem 9, we can identify

$$D_d F(X) \simeq X^{\wedge d}_{h\Sigma_d}.$$

The beginning of the Goodwillie tower for F looks like

$$\begin{tikzcd} F(X) = \Sigma^\infty\Omega^\infty X \arrow[r] \arrow[dr] & P_2F(X) \arrow[d] \\ & P_1F(X). \end{tikzcd} \tag{3}$$

Last time, we proved/argued that

$$P_1F(X) = \operatorname{colim}_n T_1^n F(X),$$

where $T_1F(X) = \Omega F \Sigma(X)$. In other words,

$$P_1F(X) = \Omega^\infty F(\Sigma^\infty X) = \Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty X;$$

but $\Omega^\infty\Sigma^\infty$ is just the identity on spectra, so

$$P_1F(X) = X.$$

Moreover, the map $F(X) \to X$ is just given by the adjoint of the identity on $\Omega^\infty X$.

Applying Ω^∞ to the tower in Equation (3) gives a splitting of the map $\Omega^\infty P_2F(X) \to \Omega^\infty X$. Since

$$D_2F(X) = (X \wedge X)_{h\Sigma_2},$$

we have a fiber sequence

$$P_2F(X) \to X \to \Sigma(X \wedge X)_{h\Sigma_2}.$$

Working 2-locally, when $X = S^{-1}$, this is (somehow, using work of Jones-Wegmann) identified with the cofiber sequence (we've implicitly applied Σ^∞ to the spaces appearing here)

$$\Sigma^{-1}\mathbf{RP}^\infty_+ \to S^{-1} \to \mathbf{RP}^\infty_{-1},$$

obtained by rotating the cofiber sequence (coming from the Kahn-Priddy transfer[6])

$$\mathbf{RP}^\infty_{-1} \to \mathbf{RP}^\infty_+ \to S.$$

By applying Ω^∞ to this discussion, we obtain the following result.

Theorem 15. *The Kahn-Priddy transfer map $\Omega Q\mathbf{RP}^\infty_+ \to \Omega QS$ admits a section.*

The statement of this theorem uses Q since we already applied Σ^∞. Note that this is not quite the Kahn-Priddy theorem, which states that $Q\mathbf{RP}^\infty_+ \to QS$ admits a section.

5. The Goodwillie tower of the identity

The identity functor on spectra has an utterly uninteresting Goodwillie tower: it is already 1-excisive. On the other hand, since pushouts are not the same as pullbacks in spaces, the identity functor on Top_* is *not* 1-excisive. Rather, we have

$$P_1(X) = \Omega^\infty\Sigma^\infty X.$$

From a moral viewpoint, this is saying that the linearization of spaces is spectra.

By Corollary 7 and the discussion in Section 2, we can construct spectra $C(n)$ such that

$$D_n(K) = \Omega^\infty \widetilde{D_n(K)} = \Omega^\infty (C(n) \wedge \Sigma^\infty K^{\wedge n})_{h\Sigma_n}.$$

Johnson-Arone-Mahowald-Dwyer have identified $C(n)$ as the Σ_n-equivariant Spanier-Whitehead dual of the classifying space of the poset of nontrivial partitions of $\mathbf{n}$. Moreover, the Goodwillie tower of this functor converges, so it remains to understand the spectra $C(n)$.

We'll see later, in Hood's and Robert's talks, the proof of the following result.

Theorem 16 (Arone-Mahowald, Arone-Dwyer)**.** *If m is an odd positive integer, then*

- *$\widetilde{D_n(S^m)} \simeq *$ if n is not a power of a prime;*
- *if $n = p^k$ for some k, then there are spectra[7] $L(k, m)$, such that*

$$\widetilde{D_{p^k}(S^m)} \simeq \Sigma^{m-k} L(k, m);$$

- *the homology $H^*(L(k, m); \mathbf{Z}/p)$ is free over the subalgebra $\mathcal{A}(k-1)$ of the mod p Steenrod algebra.*

For instance, at the prime 2,

$$L(1, n) = \Sigma^\infty \mathbf{RP}^\infty / \mathbf{RP}^n.$$

[6]At the level of spaces, there is a map $\mathbf{RP}^{n-1}_+ \to O(n)$ which sends a line to the reflection it defines; these maps are all compatible as n varies. Letting n go to ∞ and composing with the J-homomorphism gives the Kahn-Priddy map $\mathbf{RP}^\infty \to S$.

[7]These are constructed by splitting the Thom space $(B(\mathbf{Z}/p)^k)^{m\rho_k}$ (where ρ_k is the reduced real regular representation) with respect to the Steinberg idempotent living inside $\mathbf{Z}_{(p)}[\mathrm{GL}_k(\mathbf{Z}/p)]$ with respect to the obvious action of $\mathrm{GL}_k(\mathbf{Z}/p)$,