JUVITOP OCTOBER 22, 2016: THE HOPKINS-MILLER THEOREM

XIAOLIN (DANNY) SHI

Outline:

(1) Introduction: Statement of Theorem
(2) Obstruction: The Bousfield Kan Spectral Sequence
(3) Computations

Reference: Everything I'm about to say can be found in the notes by Charles Rezk. The notes are really good. It was designed to cover the theorem in one semester. And I have an hour. So...whatever you don't find from this talk, you can find there. Also, Bert Guillou has notes on the Bousfield-Kan spectral sequence. It's a very nice read. Very detailed description of the BKSS.

1. Introduction: Statement of Theorem

$\mathcal{FGL}$: the category of formal groups laws.

- Objects: (k, Γ).
- k is a perfect field of characteristic p
- Γ is a formal group law of height n over k.
- Morphism: $\alpha : (k_1, \Gamma_1) \to (k_2, \Gamma_2)$. A pair of maps $i : k_1 \to k_2$, $f : \Gamma_1 \overset{\cong}{\to} i^*\Gamma_2$.

Given a formal group law (k, Γ), you can consider its universal deformation, it is a formal group law F over the ring

$$E(k, \Gamma) = \mathbb{W}(k)[[u_1, \ldots, u_{n-1}]]$$

- Can make F 2-periodic by adding a invertible generator u in degree 2. $\Longrightarrow MU_* \to E(k, \Gamma)[u^{\pm}]$, classifying F.
- This map is Landweber exact, and so we get a homology theory, which we call $E_{(k,\Gamma)}$.
- This gives a functor

$$\begin{array}{rcl} E : \mathcal{FGL}^{\mathrm{op}} & \to & \{\text{homology theories}\} \\ (k, \Gamma) & \to & E_{(k,\Gamma)} \end{array}$$

- The $E_{(k,\Gamma)}$'s are the Lubin-Tate theories.
- It's great, because the Landweber exact functor theorem, when applied to $E(k, \Gamma)$, actually tells you a little bit more. Since the input are rings, it tells you that the homology theories are multiplicative, or the spectra representing these homology theories are homotopy commutative.

Question: Can we do better? Can we put even more structure on these spectra?

Answer: Yes. We can make them A_∞ (strictly associative), or even E_∞ (strictly commutative).

Theorem 1.1. *(Hopkins-Miller) There is a lift*

$$\begin{tikzcd} & A_\infty\text{-}ring \arrow[d, "\pi"] \\ \mathcal{FGL}^{op} \arrow[ur, dashed] \arrow[r] & Homology\ theories \end{tikzcd}$$

such that it sends

$$\begin{tikzcd} & E_{(k,\Gamma)} \arrow[d, mapsto, "\pi"] \\ (k,\Gamma) \arrow[ur, dashed, mapsto] \arrow[r, mapsto] & E_{(k,\Gamma)} \end{tikzcd}$$

The A_∞-ring spectra $E_{(k,\Gamma)}$ is also homotopy commutative and complex oriented. And if we have two formal group laws (k_1,Γ_1), (k_2,Γ_2), then

$$map_{\mathcal{A}_\infty}(E_{(k_1,\Gamma_1)}, E_{(k_2,\Gamma_2)}) \to \mathcal{FGL}((k_2,\Gamma_2),(k_1,\Gamma_1))$$

is a weak equivalence, where the LHS is a space, and the RHS is a set with discrete topology.

Consequence 1.2. We can construct higher real K-theories. Let $G \subset \mathbb{S}_n$ be a finite subgroup, we can take homotopy fixed points to construct E_n^{hG}, and they are A_∞ (or E_∞)-ring spectra as well. They are extremely important in chromatic homotopy theory, and they exist because of the Goerss-Hopkins-Miller theorem.

Consequence 1.3. Another application which is more relevant to us is, well, we can do power operations on these Lubin-Tate theories.

Method of proof:

(1) Rather than worrying about the existence of A_∞-ring structures, we should be optimistic, and assume they are A_∞ first, and then compute the space of A_∞ maps between them.
(2) To do this, use obstruction theory, by setting up the BouKanSS, spectral sequence associated to a cosimplicial object.
(3) Compute obstructions and show they vanish.
(4) Set up the obstruction for constructing A_∞-ring structures on $E_{k,\Gamma}$. Again do it by BKSS. Now, they vanish because they can be reduced to the previous obstructions, which are already computed.

So we are actually going out-of-order: we are first assuming things are A_∞-ring spectrum first, and computing the space of A_∞ maps between them. And then we go back and worry about how to put A_∞ structures on Lubin-Tate spectra. The reason is because the obstruction for computing $A_\infty(E,E)$ is essentially the same as the obstruction in computing the moduli space of A_∞ structures on E. So once we have computed $A_\infty(E,F)$, we get the uniqueness of A_∞-structures for free.

2. Obstruction: The Bousfield Kan Spectral Sequence

Goal: Let C be the (cofibrant) A_∞-operad, then we want C-alg(F, E), where F and E are A_∞ Lubin-Tate spectra. (F also needs to be cofibrant as a C-algebra spectrum)

To do so, we resolve F by the free resolution of C-algebras:

$$F \longleftarrow CF \leftleftarrows C^2F \overset{\longleftarrow}{\leftleftarrows} \cdots$$

Where CF is the free C-algebra on F, and it's weakly equivalent to

$$CF \simeq S^0 \vee F \vee (F \wedge F) \vee \cdots$$

We are interested in dual object, so consider the cosimplicial object $Y^\bullet$, defined by

$$Y^n = C\text{-alg}(C^{n+1}F, E)$$

Well, what do they do? Associated to $F_\bullet$ is a skeletal filtration of the geometric realization

$$|F_\bullet|_0 \to |F_\bullet|_1 \to \cdots \to |F_\bullet|$$

And dually, associated to $Y^\bullet$ is the Tot-tower – a tower of fibrations

$$\mathrm{Tot}^0(Y^\bullet) \leftarrow \mathrm{Tot}^1(Y^\bullet) \leftarrow \mathrm{Tot}^2(Y^\bullet) \leftarrow \cdots \leftarrow \mathrm{Tot}(Y^\bullet)$$

An important fact is that in our case, this Tot-tower is dual to the filtration for the geometric realization:

$$\text{C-alg}(|F_\bullet|_0, E) \leftarrow \text{C-alg}(|F_\bullet|_1, E) \leftarrow \text{C-alg}(|F_\bullet|_2, E) \leftarrow \cdots \leftarrow \text{C-alg}(|F_\bullet|, E)$$

Why is this fact useful? Well, in our case, the natural map $|F_\bullet| \to F$ is an equivalence of C-algebras. So $\mathrm{Tot}(Y^\bullet)$ *is* C-alg(F, E). Computing the Tot-tower is what we want.

This is exactly what the Bousfield-Kan spectral sequence does. It's the spectral sequence associated to this Tot-tower (which, in our case, is a tower of fibrations). The E_2-page (f is the base point) can be identified by

$$E_2^{s,t} = \pi^s \pi_t Y^\bullet \Longrightarrow \pi_{t-s}(\mathrm{Tot}\, Y^\bullet, f)$$

Where $E_2^{*,t}$ is the cohomotopy of the cosimplicial object

$$\pi_t Y^0 \rightrightarrows \pi_t Y^1 \overset{\rightarrow}{\underset{\rightarrow}{\leftleftarrows}} \pi_t Y^2 \cdots$$

(with respect to the base point determined by f). The differentials are

$$d_r : E_r^{s,t} \to E_r^{s+r, t+r-1}$$

I have to say, this is the most confusing spectral sequence that I have ever seen, it probably doesn't look very bad, but as you look at it more, there are a lot of places where it's confusing. So I would like to take some time to explain them.

(1) The spectral sequence is "fringed". When
- $t \geq 2$, we can compute the cohomotopy by computing the cohomology of the corresponding co-chain complex. They are abelian groups.
- When $t = 1$, we are computing with π_1, and unless we have a loop space, we would only get groups, not necessarily abelian.

- When $t = 0$, there is a corner,

$$E_2^{0,0} = \mathrm{Ker}(\pi_0 Y^0 \rightrightarrows \pi_0 Y^1)$$

this equalizer is only a pointed set.
- (Draw a diagram)

(2) To define the spectral sequence and say what its E_2 terms are, I need to choose a base point for $\mathrm{Tot}(Y^\bullet)$. Suppose we already know a C-algebra map $f : F \to E$, then this makes our cosimplicial space pointed, so it will give us a base point for $\mathrm{Tot}\, Y^\bullet$. So the spectral sequence exists and there is no problem.

(3) Except there is a problem. We need a C-algebra map from $F \to E$ to start with. This is not trivial. And in fact, when we move on to trying to put an A_∞ structure on E, that is the whole problem. An A_∞ structure corresponds to a map of operads from $A_\infty \to \mathcal{E}_X$, and all we want is just one point. So we need a starting point. And getting a base point for the totalization $\mathrm{Tot}(Y^\bullet)$ is highly nontrivial.

(4) To solve this existential crisis, we would like to find a base point for $\mathrm{Tot}(Y^\bullet)$. It turns out we can do this in stages.
- Choose a base point in $\mathrm{Tot}^0 Y^\bullet = Y^0$ (that's something in $E_1^{0,0}$).
- Want to start lifting up the Tot-tower $\Leftrightarrow$ does it survive to the E_∞-page?
- Obstruction: possible differentials targeting things in the (-1)-stem.
- "Obstruction lives in the spectral sequence". (Which could be a little brain-fuck, because at first, it seems like the spectral sequence won't even exist if we don't have a base point, but yet the spectral sequence is going to tell us about the existence of the base point. It's like the chicken and the egg).
- The point is, the obstructions are not ALL there at the same time. The obstruction at each stage only occur after we have solved the obstruction for the previous stage, and it actually depends on the choice of lifting we made on the previous stage.
- So to summarize:
  (a) To lift to $\mathrm{Tot}^1 Y^\bullet$, we need our point to survive to $E_2^{0,0}$.
  (b) To lift to $\mathrm{Tot}^2 Y^\bullet$, we need the point to survive to $E_3^{0,0}$, meaning that the obstruction in $E_2^{2,1}$ vanishes.
  (c) To lift to Tot^3, need the obstruction in $E_3^{3,2}$ vanish.
  (d) $\cdots$
- In our case, we are very lucky, because as we are doing the computation, it turns out that $E_2^{s,s-1} = 0$ for all $s \geq 2$ (in fact, you see 0 everywhere except at one place). So we are in the dream world where everything degenerates.

3. Computations

Now that we have the spectral sequence, we gotta compute the E_2-page. It turns out that the spectral sequence collapses, and it's 0 everywhere except for one place, $E_2^{0,0}$.

Recall we are trying to compute C-alg(F, E), where E is $E_{(k_1,\Gamma_1)}$, and F is $E_{(k_2,\Gamma_2)}$.

Theorem 3.1. *(the most important computational result in the paper) The E_2 terms are identified as follows:*

$$\begin{aligned} E_2^{0,0} &\cong Hom_{E_*\text{-}alg}(E_*F, E_*) \\ E_2^{s,t} &\cong Der^s_{E_*\text{-}alg}(E_*F, E_{*+t}), \text{ for } t-s \geq -1, \text{ and } s,t \geq 0 \end{aligned}$$

After computation, we get

$$\begin{aligned} Hom_{E_*\text{-}alg}(E_*F, E_*) &\cong \mathcal{FGL}((k_1,\Gamma_1),(k_2,\Gamma_2)) \\ Der^s_{E_*\text{-}alg}(E_*F, E_{*+t}) &\cong 0, \quad \text{for all } s,t \geq 0 \end{aligned}$$

We will be content on doing the first computation. Something general to keep in mind is that these computations requires a lot of knowledge and tricks of working with Landweber exact spectra and Lubin-Tate theories. Here is a slogan:

Slogan : "Landweber exact theories are "flat" $\Longrightarrow$ have a lot of good properties. Lubin Tate theories are even better. "

In particular, here are some highlight facts: if E and F are Landweber exact, then

(1) E_*F is a flat E_*-module.
(2) Kunneth SS & universal coefficient SS:

$$\mathrm{Tor}_s^{E_*}(E_*X, E_{*+t}Y) \Longrightarrow E_{s+t}(X \wedge Y)$$

$$\mathrm{Ext}^s_{E_*}(E_*X, E_{*+t}) \Longrightarrow E^{t-s}X$$

(3) Consequence: if E and F are Lubin-Tate spectra, then

$$E^*F \to \mathrm{Hom}_{E_*}(E_*F, E_*)$$

is an isomorphism (can show that higher Ext terms vanish).

3.1. Identification of $E_2^{0,0}$.

we will show that

$$E_2^{0,0} \simeq \mathrm{Hom}_{E_*\text{-alg}}(E_*F, E_*)$$

From our discussion of the Bousfield-Kan spectral sequence, $E_2^{0,0}$ is the equalizer of

$$\pi_0 C\text{-alg}(CF, E) \rightrightarrows \pi_0 C\text{-alg}(C^2F, E)$$

To compute this equalizer, we gotta replace it with something more reasonable (aka more algebraic) to work with.

Let Y be any C-algebra. There is a map

$$\pi_0 C\text{-alg}(Y, E) \to \mathrm{Hom}_{E_*\text{-alg}}(E_*Y, E_*)$$

Defined by taking $E_*(-) : E_*Y \to E_*E \to E_*$. The A_∞ structure of Y gives E_*Y the structure of an associative E_*-algebra.

For our case, $Y = CF$. We would like to identify

$$\mathrm{Hom}_{E_*\text{-alg}}(E_*(CF), E_*)$$

At the level of the homotopy category of spectra,

$$CF \simeq S^0 \vee F \vee (F \wedge F) \vee \cdots$$

Algebraically, there is a construction that's related to this. Let M be a E_*-module. Then let TM be the E_*-tensor algebra on M:

$$TM := E_* \oplus M \oplus (M \otimes_{E_*} M) \oplus \cdots$$

- Start: inclusion map $X \to CX$.
- $\overset{induces}{\Longrightarrow}$ $E_*X \to E_*(CX)$ of E_*-modules.
- $\overset{induces}{\Longrightarrow}$ $T(E_*X) \to E_*(CX)$ of E_*-algebras.
- By the flatness hypothesis, easy to see if E_*X is flat over E_*, then $E_*(CX) \simeq T(E_*X)$.
- This holds when $X = CF, C^2F, \ldots$ because F is Landweber exact.

Consider the following diagram:

$$\begin{tikzcd}
\pi_0 C\text{-alg}(CX, E) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{E_*\text{-alg}}(E_*(CX), E_*) \arrow[d, "\simeq"] \\
{[X, E]} \arrow[r, "\simeq"] & \mathrm{Hom}_{E_*}(E_*X, E_*)
\end{tikzcd}$$

The maps are defined in the obvious way.

- Left vertical iso: CX is the free C-algebra on X.
- Bottom horizontal iso: E_*X is flat over E_*. And we have Lubin-Tate spectra all over the place. So higher Ext terms vanish in universal coefficient theorem.
- Right vertical iso: $E_*CX = T(E_*X)$, and $\mathrm{Hom}_{E_*\text{-alg}}(T(E_*X), E_*) \simeq \mathrm{Hom}_{E_*}(E_*X, E)$.
- $\Longrightarrow$ Top horizontal arrow an iso.

Before:

$$\pi_0 C\text{-alg}(CF, E) \rightrightarrows \pi_0 C\text{-alg}(C^2F, E)$$

After:

$$\mathrm{Hom}_{E_*\text{-alg}}(T(E_*F), E_*) \rightrightarrows \mathrm{Hom}_{E_*\text{-alg}}(T^2(E_*F), E_*)$$

This reduces to computing the equalizer of

$$\mathrm{Hom}_{E_*}(E_*F, E_*) \rightrightarrows \mathrm{Hom}_{E_*}(T(E_*F), E_*)$$

(Notice that E_*F and E_* actually have the structure of an E_*-algebra, but we are not requiring the Hom-set to be over E_*-algebras.)

Given a map $f : E_*F \to E_*$ in the source, the top map sends it to the composite

$$T(E_*F) \to E_*F \xrightarrow{f} E_*$$

and the bottom map sends it to the composite

$$T(E_*F) \xrightarrow{T(f)} T(E_*) \to E_*$$

The equalizer of these two maps is exactly the set of E_*-algebra maps from $E_*F \to E_*$. Therefore

$$E_2^{0,0} = \mathrm{Hom}_{E_*\text{-alg}}(E_*F, E_*).$$