SUVITOP 01/04/15

p-adic htpy theory

Def: A space X is p-finite if:

- $\pi_0 X$ is non $\emptyset$ & finite
- $\pi_i(X,x)$ is always a finite p-group
- $\pi_i(X,x)=0$ for $i \gg 0$

Ex: · Eilenberg-MacLane spaces of finite p-groups.
· finite Postnikov tower

Prop: If X is a connected p-finite space there's a tower of fibrations

$$X \simeq X_n \to X_{n-1} \to \dots \to X_1 \to X_0 = *$$

for which $\mathrm{fib}(X_i \to X_{i-1}) = K(\mathbb{F}_p, j)$

Proof: X admits a Postnikov tower & the finite p-groups are solvable & we can lift the filtration of $\pi_i X$ to a filtr of $P^i X$. To do that we need to find a filtration invariant under the action of $\pi_1 X$. We can always do that because e is fixed under the action $\Rightarrow$ there must be another element.

(So $\pi_n X = H_r \supsetneq H_{r-1} \supsetneq \dots \supsetneq H_0 = 1$ $[H_i : H_{i-1}] = p$, H_i stable under π_1) $\square$

Prop: If X is p-finite then $H^n(X;\mathbb{F}_p)$ are finite v. spaces

Proof: Check it for $X = K(\mathbb{F}_p, n)$ & then use the Serre SS to induct. $\square$

Let $S_{p\text{-fin}} \subseteq S$ full subcategory of spaces spanned by p-finite spaces.

Thm (Mandell) The functor $Y \longmapsto C^*(Y; \overline{\mathbb{F}}_p)$ is a fully faithful embedding of $S_{p\text{-fin}}$ into commutative $\overline{\mathbb{F}}_p$-algebras.

Def: $S_{p\text{-profin}} = \mathrm{Pro}(S_{p\text{-fin}})$

Remark: The limit functor $S_{p\text{-profin}} \to S$ admits a left adjoint L.

Prop: The composition $X \to \lim L X =: \hat{X}$ can be computed by the formula
$$\hat{X} = \lim_{\substack{X \to Q \\ Q \in \mathcal{S}_{p\text{-fin}}}} Q$$

Thm: Let X be a simply connected space w/ $\pi_i X$ fg as a $\mathbb{Z}$-mod. Then
$$X \to \hat{X}$$
is the p-completion (e.g. $\pi_i X \otimes \mathbb{Z}_p \xrightarrow{\sim} \pi_i \hat{X}$)

Proof: ① We prove it for E-ML spaces $K(\mathbb{Z}, n)$, $K(\mathbb{Z}/\ell, n)$
② Conclude w/ the Eilenberg-Moore ss $\leftarrow$ much easier given Mandell's thm

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Cochains on p-(pro)finite spaces

Recall: Fix a field k, there's a symm. mon. ∞-cat. $\mathcal{D}(k)$
For any space X define
$$\mathrm{Loc}(X) = \mathrm{Fun}(X, \mathcal{D}(k))$$
If $\mathcal{L}$ is a local system $C^*(X; \mathcal{L}) = \lim_X \mathcal{L}$
If $\mathcal{L}$ is the constant local syst. $C^*(X; \mathcal{L}) =: C^*(X; k)$ has a natural E_∞-algebra structure.

Def: We define $C^*(-; k)$ on $\mathcal{S}_{p\text{-profin}}$ by asking that it sends filt. limits to filtered colims.

Thm: Let k be an algebraically closed field of char p
$$X \longmapsto C^*(X; k)$$
is a fully faithful functor $\mathcal{S}_{p\text{-profin}} \to \mathrm{CAlg}_k$.

Thm: Let k be a field of char p. Then the functor $S_{p\text{-profin}} \xrightarrow{F} CAlg_k$

$$F: X \mapsto C^*(X; k)$$

sends limits to colimits

Proof: We defined F so that it respects finite limits. It's obvious it sends $*$ to k. We just need to show it sends pullbacks to pushouts.

Suppose $\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$ is pullback square. We want that $\begin{array}{ccc} C^*A & \leftarrow & C^*B \\ \uparrow & & \uparrow \\ C^*C & \leftarrow & C^*D \end{array}$.

We can assume wlog A, B, C, D are p-finite. ✓

When $D = *$ this is the Eilenberg-Zilber thm. ✓

Let $\mathcal{L}_A, \mathcal{L}_B, \mathcal{L}_C$ the local systems on D by sending

$$d \mapsto C^*(A_d; k),\ C^*(B_d; k),\ C^*(C_d; k)$$

Then $\mathcal{L}_A \cong \mathcal{L}_B \otimes \mathcal{L}_C$ by E-Z thm.

Lemma: Let D be p-finite & suppose $\mathcal{L}_0, \mathcal{L}_1$ are two local systems s.t. $H^i(D, \mathcal{L}_j) = 0$ for $i < 0$. Then the canonical map

$$C^*(D, \mathcal{L}_0) \otimes_{C^*(D)} C^*(D, \mathcal{L}_1) \xrightarrow{\sim} C^*(D; \mathcal{L}_0 \otimes \mathcal{L}_1)$$

IDEA: · Examine the $\mathcal{L}_0$ for which this holds $\forall \mathcal{L}_1$. This is closed under filtered colims & extensions of local system

- We can assume $\mathcal{L}_0$ is concentrated in a single degree & finite dimensional $\Rightarrow$ it is a fd repr. of $\pi_1 X$. We can assume it is irreducible. Since $\pi_1 X$ is a p-group $\Rightarrow \mathcal{L}_0$ is trivial. Then it's obvious. □

Sketch of the proof of Mandell's thm:

For Y, X p-profinite

$$\mathrm{Hom}(Y, X) \xrightarrow{\overset{?}{\sim}} \mathrm{Hom}(C^*X, C^*Y)$$

Both sides preserve limits in $X \Rightarrow$ we can assume $X = K(\mathbb{F}_p, j)$.
We need to show $\forall i$ $\exists$ an equivalence

$$H^{n-i}(Y;\mathbb{F}_p) \longrightarrow \pi_i \operatorname{Hom}\left(C^*(K(\mathbb{F}_p,n)), C^*(Y;\mathbb{F}_p)\right)$$

<u>Thm</u> (Mandell) Let F denote the free E_∞-alg $/k$ generated by a class η in degree $-n$ $\Rightarrow$ there's a cofiber sequence

$$F \xrightarrow{\partial} F \longrightarrow C^*(K(\mathbb{F}_p,n), k)$$

$$\partial = \eta - P^0\eta$$

⚠ This is very hard!

Having this, in the end everything boils down to understand

$$\begin{array}{c} H^{n-i}(Y;k) \xrightarrow{\mathrm{id}-P^0} H^{n-i}(Y;k) \\ \| \\ H^*(Y;\mathbb{F}_p)\otimes_{\mathbb{F}_p} k \end{array}$$

?

P^0 acts by id on the left & the Frobenius on the right

$\Rightarrow$ We are done: this is surj w/ kernel $H^*(Y;\mathbb{F}_p)$.

<u>Remark</u>: If k is not algebraically closed

$$\mathrm{CAlg}_k \simeq \text{Sheaves in } p\text{-profinite spaces on the étale site of } \operatorname{Spec} k$$

Recall that in rational homotopy theory every commutative $\mathbb{Q}$-algebra is cdga, we have minimal models & formality
Can we have any of this?

- No hope for strictification (due to the Steenrod operations)
- Mandell's calculation can be viewed as giving a minimal model for $C^*(K(\mathbb{F}_p,j))$ & there's a good theory of minimal model.

Def: A comm. algebra $/\overline{\mathbb{F}}_p$ is formal if it's equivalent to its cohomology

Prop: The only formal spaces in this sense are contractible.

Def: An E_n-algebra in $\mathcal{D}(\overline{\mathbb{F}}_p)$ is formal if it's equivalent to its cohomology.

Prop: For any space X, $\Sigma^n X$ is E_n-formal. In particular S^n is E_n-formal & not E_{n+1}-formal.

Conj: Let X be an n-connected p-complete space. Then $C^*(X;\overline{\mathbb{F}}_p)$ is equivalent as an E_n-algebra to a CDGA. If X is rationally formal it is also E_n-formal.