JUVITOP 18/03/15

Deformation quantisation

Def: A <u>classical system</u> is a manifold M (<u>phase space</u>). The algebra of observables a subalgebra $C^\infty(M)$. Time evolution is governed by the Poisson bracket

$$\{\,,\,\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$$

Def: A <u>quantum system</u> is an Hilbert space H, the algebra of observables is an algebra $\mathcal{O}(H)$ of operators on H & time evolution is governed by $[\,,\,]$

In quantum mechanics we have the Planck constant $\hbar$. Let's regard it as a formal parameter. Observables in QM often commute mod $\hbar$.

$$[S,T] = \sum_{n\geq 1} c_n(S,T)\,\hbar^n$$

Let M be a Poisson manifold, a <u>deformation quantisation</u> of M is an associative product $*: C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$

$$f * g = B_0(f,g) + B_1(f,g)\,\hbar$$

where
$$B_0(f,g) = fg$$
$$B_1(f,g) = \{f,g\}$$

Thm (Kontsevich) Deformation quantisation exist functorially for Poisson manifold

Def: A <u>formal moduli problem</u> is a functor

augm. f.d. local $\mathbb{C}$-alg $\rightarrow$ $\mathrm{Art}_{\mathbb{C}} \longrightarrow \mathrm{Gpd}$

which looks like it should be representable.

Ex: Let A be an associative $\mathbb{C}$-algebra, $\forall a \in \mathrm{Art}_{\mathbb{C}}$

$\mathrm{Def}_A(a)$ = gpd of associative a-algebras $\tilde{A}$ w/ an iso $\tilde{A} \otimes_a \mathbb{C} \xrightarrow{\sim} A$.

<u>Def</u>: Let $\mathfrak{g}$ be a graded v. space,

$T_{\mathfrak{g}} := \mathbb{C}[[\mathfrak{g}^\vee[-1]]]$ ("infinitesimal nbhd of 0 in $\mathfrak{g}[1]$")

We'll consider vector fields on Spf $T_{\mathfrak{g}}$ D of degree -1 which are 0 at 0

Such a v. field is a continuous derivation

$$\hat{q}: T_{\mathfrak{g}} \to T_{\mathfrak{g}}[-1]$$

This corresponds to $Q: T_{\mathfrak{g}}^\vee \to T_{\mathfrak{g}}^\vee[+1]$ (cofree cocomm. coalg on $\mathfrak{g}$)

$$T_{\mathfrak{g}}^\vee = \bigoplus_{n\geq 0} (\mathfrak{g}[1]^{\otimes n})^{\Sigma_n}$$

Q must be a coderivation.

So it is enough to give a linear map $T_{\mathfrak{g}}^\vee \to \mathfrak{g}[2]$

This, in turn, is the datum of linear maps

$P_1 : \mathfrak{g}[1] \to \mathfrak{g}[2]$ (degree 1 endomorphism)

$P_2 : \mathfrak{g}[2]^{\otimes 2\, \Sigma_2} \to \mathfrak{g}[2]$ (degree 0 bracket, antisymmetric)

Suppose further that $[D, D] = 0$, then

- $P_1^2 = 0$
- P_2 satisfies the Jacobi identity up to P_3
- Leibniz rule $P_1 P_2 = P_2(P_1 \otimes 1) + P_2(1 \otimes P_1)$

<u>Def</u>: A dgla is a graded v. space $\mathfrak{g}$ together w/ a degree 1 self commuting v. field on Spf $T_{\mathfrak{g}}$ s.t. $P_i = 0$ $\forall i > 2$.

<u>Def</u>: An L_∞-algebra is a graded v. space $\mathfrak{g}$ together w/ a degree 1 self commuting v. field on Spf $T_{\mathfrak{g}}$

We can get a formal stack from an L_∞-algebra which is the 0-set of D on Spf $T_{\mathfrak{g}}$.

Of course every formal stack represents a formal moduli problem.

Thm (Pridham, Lurie): Once everything is derived enough this gives an equivalence of categories & formal moduli problem.

Ex: Let A be an associative $\mathbb{C}$-algebra. The cohomological Hochschild complex is

$$C^*(A,A) = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A)$$

$$(df)(a_1,\dots,a_{n+1}) = a_1 f(a_2,\dots,a_{n+1}) \pm \sum_{i=1}^{n} f(a_1,\dots,a_i a_{i+1},\dots,a_{n+1})$$
$$\pm f(a_1,\dots,a_n)a_{n+1}$$

Suppose $f \in C^n(A,A)$, $g \in C^m(A,A)$

$$(f\circ g)(a_1,\dots,a_{m+n-1}) = \sum_{i=1}^{n} f(a_1,\dots,a_{i-1}, g(a_i,\dots,a_{i+m-1}), a_{i+m},\dots,a_{n+m-1})$$

$$[f,g] = f\circ g - g\circ f$$

$\Rightarrow C^*(A,A)$ is a dgla. What is the formal moduli problem?

This is the formal scheme of deformations of A.

Ex: Let R be a commutative ring. Let $PV^*(R)$ be the algebra of polyvector fields on Spec R

$$PV^n(R) = \langle v_1 \wedge \dots \wedge v_n \rangle \quad v_i \text{ v. field on Spec} R$$

DGLA w/ 0-differential

Schouten-Nijenhuis bracket

$$[v_1\wedge\dots\wedge v_n, w_1\wedge\dots\wedge w_m] = \sum_{i,j} [v_i, w_j]\wedge v_1\wedge\dots\wedge\hat{v}_i\wedge\dots\wedge v_n\wedge w_1\wedge\dots\wedge\hat{w}_j\wedge\dots\wedge w_m$$

This DGLA gives rise to the formal scheme of deformations of the 0 Poisson bracket on R

Goal: Show that these are equivalent deformation theories.

Any def of R gives a deform. of the zero Poisson bracket on R
$\Rightarrow \hbar\{,\}$

<u>Step 1</u>: Show that $C^*(R,R)$ is an E_2-algebra
(Deligne conjecture). The formality of the E_2-operad shows
$C^*(R,R) \cong HH^*(R)$ as an E_2-algebra.
<u>Step 2</u>: We know that $HH^*(R)$ is equivalent as a complex to polyvector fields on R (HKR thm)
We want to upgrade it to an equivalence of E_2-algebras
This only works when $R = \mathbb{C}[x_1, \dots, x_n]$,

"A solution of the Deligne conjecture" McClure - Smith
<u>Def</u>: An operad w/ multiplication is an operad $\mathcal{O}$ w/ a morphism w/ the A_∞-operad (picks out a distinguished μ in $\mathcal{O}(2)$)
Then $\mathcal{O}(n)$ fit together to form a cosimplicial space

$\nu \in \mathcal{O}(n)$

$d^i \nu = \nu \circ_i \mu$

$s^i \nu = \nu \circ_i 1$

<u>Ex</u>: A be an associative ring. A has an endomorphism operad $End(A)$
Then $\boxed{\mathrm{Tot}\ End(A)_\bullet = C^*(A,A)}$

<u>Idea</u>: Define an operad in posets which have
ⓐ models E_2
ⓑ acts on $\mathrm{Tot}\ \mathcal{O}(\cdot)$ for any operad w/ mult.

<u>Def</u>: The set $W(n)$ of words of length n is the set of expressions in the symbols $\{1, \dots, n\}$ built from formal multipl. & composition

Ex: $1(5(3), 2, 4 \cup 6) \in W(6)$

$\{W(n)\}$ form (almost) an operad structure w/

$g = 1(2 \cup 3) \in W(6)$
$\quad 5(6 \cup 7)$

$f \circ_5 g = 1(5(6 \cup 7, 3), 2, 4 \cup 8) \in W(8)$

There's a partial order on each $W(n)$ generated by

- $1(2, \dots, n) > 2 \cup 1(3, \dots, n)$
- $1(2, \dots, n) > 1(2, \dots, n-1) \cup n$
- $1(2, \dots, n) > 1(2, \dots, i \cup i+1, \dots, n)$

& compatible w/ the operad structure.

$W(2)$

$$\begin{tikzcd}
& 1(2) \arrow[dl] \arrow[dr] & \\
1 \cup 2 & & 2 \cup 1 \\
& 2(1) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Circle!