Pre-Talbot Seminar, March 18, 2015: Deformation Quantization for Topologists

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1 Introduction

The motivation for deformation quantization comes from the problem of passing from classical to quantum mechanics. Let's recapitulate some standard formalizations of these two situations:

Definition 1.1. A *classical system* is a finite-dimensional Poisson manifold M (the *phase space* of the system). The *algebra of observables* of the system (e.g. position, momentum) is the algebra $C^\infty(M)$ of complex-valued smooth functions on M, and the time evolution of observables is governed by the *Poisson bracket*, which is an antisymmetric bilinear form

$$\{,\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$$

satisfying a distributive law with respect to the product.

Definition 1.2. A *quantum system* is a Hilbert space $\mathcal{H}$ (the *space of states* of the system). The *algebra of observables* of the system is the algebra $\mathcal{O}(\mathcal{H})$ of (nice) linear operators on $\mathcal{H}$, and the time evolution of observables is governed by the commutator bracket $[,]$ of operators.

Here's a question of obvious physical interest: given a classical system, can we define a corresponding quantum system in such a way that the physical meaning of the observables is preserved?

The first thing one might try is to assign quantum observables to classical observables in such a way that the Poisson bracket maps to the commutator. Unfortunately, it was shown in the early 20th century that this isn't possible.

There's a fundamental physical quantity in quantum mechanics called Planck's constant and denoted $\hbar$. One can regard $\hbar$ as a formal parameter and observe that observables in quantum mechanics tend to commute "mod $\hbar$":

$$[S, T] = C_1(S, T)\hbar + C_2(S, T)\hbar^2 + \cdots$$

The idea of deformation quantization is to identify the Poisson bracket with C_1, the "linear part" of the commutator.

Definition 1.3. Let M be a Poisson manifold. A *deformation quantization* of M is an associative product

$$\star : C^\infty(M)[[\hbar]] \otimes C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$$

such that, writing

$$f \star g = B_0(f,g) + B_1(f,g)\hbar + B_2(f,g)\hbar^2 \cdots$$

we have $B_0(f,g) = fg$ and $B_1(f,g) = \{f,g\}$.

(Then $C_1(f,g) = 2B_1(f,g)$, by antisymmetry.)

Remark 1.4. Really, there's a groupoid of deformation quantizations of M whose morphisms are reparametrizations by continuous automorphisms of $\mathbb{C}[[\hbar]]$ which preserve B_0 and B_1, but we'll elide this here.

Remark 1.5. There's a sneaky red herring here: the reason we're allowed to stipulate $B_1 = \{f,g\}$ rather than some arbitrary bilinear form such that $B_1(f,g) - B_1(g,f) = 2\{f,g\}$ is that any formal deformation of $C^\infty(M)$ to an associative algebra can be reparametrized into a deformation where B_1 is anticommutative.

Kontsevich constructed a functorial deformation quantization for Poisson manifolds. As we'll see, this apparently innocuous algebraic problem leads to some surprisingly deep mathematics. In this talk, we'll only discuss Kontsevich's result when the underlying manifold is Euclidean space $\mathbb{R}^n$, because that's hard enough.

2 Deformation theory

We're deforming today, so let's recall a few concepts from deformation theory.

Definition 2.1. A *formal moduli problem* is a functor from the category **Art** of augmented finite-dimensional local commutative $\mathbb{C}$-algebras to groupoids that looks like it's supposed to be representable by some kind of stack; I won't go into the details of the definition here.

Example 2.2. Fix an associative algebra A. Define a functor $\mathbf{Def}_A : \mathbf{Art} \to \mathbf{Gpd}$ by letting $\mathbf{Def}_A(\mathfrak{a})$ be the groupoid whose objects are associative $\mathfrak{a}$-algebras $\widetilde{A}$ together with isomorphisms

$$\phi : \widetilde{A} \otimes_{\mathfrak{a}} \mathbb{C} \xrightarrow{\sim} A$$

and whose morphisms are isomorphisms of $\mathfrak{a}$-algebras commuting with ϕ.

Most examples of formal moduli problems I've come across are of pretty much this flavor. As you can see, this one in particular has a rather direct bearing on the problem at hand.

You'll often hear the phrase "deformation theory is controlled by differential graded Lie algebras" uttered by the literati. I find this infuriatingly vague, but it's a shorthand for a very concrete truth that can be thought of as the fundamental theorem of deformation theory. So let's set up the somewhat weird presentation of DGLAs that's required to tell this story. I don't really know how to motivate this, so I'll just tell it like it is.

Let $\mathfrak{g}$ be a graded vector space. We're going to shift it up by 1 and take the dual - now we have $\mathfrak{g}^\vee[-1]$. Let's take the free complete commutative algebra on this,

$$T_\mathfrak{g} := \mathbf{C}[[\mathfrak{g}^\vee[-1]].$$

We'll consider vector fields on the derived formal scheme Spf $T_\mathfrak{g}$, which should be thought of as a formal neighborhood of the origin in $\mathfrak{g}[1]$. In particular, we'll consider vector fields D of degree -1 (why not?) which are zero at the origin.

What is such a vector field? Algebraically, it's a continuous derivation

$$\hat{Q} : T_\mathfrak{g} \to T_\mathfrak{g}[-1]$$

which is compatible with the augmentation. We're looking for equations in $\mathfrak{g}$, and $T_\mathfrak{g}$ is built from $\mathfrak{g}^\vee$, so let's just take the $\mathbb{C}$-linear dual of everything; the dual $T_\mathfrak{g}^\vee$ of $T_\mathfrak{g}$ is the cofree coalgebra on $\mathfrak{g}[1]$, and $\hat{Q}$ corresponds to a coderivation

$$Q : T_\mathfrak{g}^\vee \to T_\mathfrak{g}^\vee[1].$$

For this, it suffices to give a linear map

$$P : T_\mathfrak{g}^\vee \to \mathfrak{g}[2],$$

maybe satisfying some conditions, because then we can extend by the co-Leibniz rule. This, in turn, is the data of linear maps

$$P_1 : \mathfrak{g}[1] \to \mathfrak{g}[2], P_2 : (\mathfrak{g}[1]^{\otimes 2})^{\Sigma_2} \to \mathfrak{g}[2], P_3 : (\mathfrak{g}[1]^{\otimes 3})^{\Sigma_3} \to \mathfrak{g}[2], \cdots$$

Observe that P_1 is a degree 1 endomorphism of $\mathfrak{g}$, and P_2 is a degree 0 antisymmetric product on $\mathfrak{g}$.

Finally, assume that Q commutes with itself:

$$[Q, Q] = 0.$$

This condition is of course trivial for vector fields of even degree, but Q is odd so anything can happen. At the level of the P_i, this shows that $P_1^2 = 0$, so that P_1 is a differential on $\mathfrak{g}$, and that P_2 satisfies a Jacobi identity; the coderivation condition implies that P_1 is a derivation with respect to P_2. We've arrived at the following unconvential definition:

Definition 2.3. A differential graded Lie algebra is a graded vector space $\mathfrak{g}$ together with a degree 1, self-commuting vector field on $T_\mathfrak{g}$ such that $P_i = 0$ for $i > 2$.

Emboldened by our success, we go further:

Definition 2.4. An L_∞-algebra is a graded vector space $\mathfrak{g}$ together with a degree 1, self-commuting vector field.

This definition behaves the way you'd expect it to. For instance, once you have P_3, you no longer have the Jacobi identity strictly, but P_3 provides a cell whose boundary is the Jacobi identity. I get very confused by this, because it seems to be putting the differential - which I think of as part of the fabric of the underlying object - on the same footing as the Lie bracket, which is a structure on the object. But that's life.

So what's the point? This definition makes it clear how to associate a formal scheme, and hence a formal moduli problem, to a DGLA of L_∞-algebra - we just take the formal schemes of zeroes of the vector field D. (It's actually a formal stack, once we take into account certain automorphisms of $T_{\mathfrak{g}}$.)

Theorem 2.5. (Pridham, Lurie, ...) Once everything is sufficiently derived, this assignment gives an equivalence of categories between L_∞-algebras and formal moduli problems.

We're going to start writing down lots of formulae. For the sake of sanity, we'll suppress most signs.

Example 2.6. Let A be an associative ring and consider the cohomological Hochschild complex $C^*(A, A)$. We have

$$C^n(A, A) = \mathrm{Hom}(A^{\otimes n}, A)$$

and if $f \in C^n(A, A)$, then

$$df(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_{n+1}) + f(a_1 \otimes \cdots \otimes a_n) a_{n+1}.$$

We can make $C^*(A, A)$ into a DGLA: if $f \in C^n(A, A)$ and $g \in C^m(A, A)$, we define $f \circ g \in C^{n+m-1}(A, A)$ by

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{n+m-1}) = \sum_{i=1}^{n} f(a_1 \otimes \cdots g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes \cdots a_{n+m-1}).$$

and

$$[f, g] = f \circ g - g \circ f.$$

This DGLA gives rise to the formal scheme of deformations of A as an associative algebra.

Example 2.7. Let R be a commutative ring and let $PV^*(R)$ be the algebra of polyvector fields on Spec R. We have a Lie bracket, the Schouten-Nijenhuis bracket, on $PV^*(R)$ given by

$$[v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_m] = \sum_{i,j} ([v_i, w_j] \wedge \{\text{everything else}\}).$$

With the zero differential, this becomes a DGLA which gives rise to the formal scheme of deformations of the zero Poisson bracket on R (so the $\mathfrak{a}$-points are Poisson structures on $R \otimes \mathfrak{a}$ which vanish on the special fiber).

At this point the strategy becomes clear. If we can give an equivalence of L_∞-algebras between $C^*(R, R)$ and $PV^*(R)$, we'll have shown that they represent the same formal moduli problem: the problem of deforming the zero Poisson bracket to a Poisson bracket on $R[[\hbar]]$[1] is the same as the problem of giving an associative multiplication on $R[[\hbar]]$ agreeing with the multiplication on R to zeroth order. But the former deformation problem has a canonical solution: if $\{,\}$ is a Poisson bracket on R, then $\hbar\{,\}$ is a deformation of the zero Poisson bracket. We just have to check that this correspondence can be made compatible with linear terms, and we'll be home safe.

This program will be carried out in two steps. First we'll show that $C^*(R, R)$ is an algebra over the E_2-operad in a way compatible with the Lie bracket; this is the vaunted Deligne conjecture. By the formality of the E_2-operad discussed by Umut last week, we'll have shown that $C^*(R, R)$ is equivalent to its cohomology algebra $HH^*(R, R)$ as an E_2-algebra.

By the Hochschild-Kostant-Rosenberg theorem, $HH^*(R, R)$ is isomorphic to $PV^*(R)$ as a complex. The final step is to lift this to an equivalence of E_2-algebras. This step only works when $R \cong \mathbb{R}[x_1, \cdots, x_n]$ is a polynomial algebra, and we probably won't say much about it, because Kontsevich is pretty vague about how it works.

3 The Deligne conjecture

There are now as many proofs of the Deligne conjecture as there are stars in the sky. Our sketch will loosely follow the paper "A solution of Deligne's conjecture" by McClure and Smith, which has the bonus of giving a cool combinatorial model for the E_2 operad.

Definition 3.1. An *operad with multiplication* is an operad $\mathcal{O}$ together with a map of operads $A_\infty \to \mathcal{O}$.

This just means that $\mathcal{O}$ has some distinguished binary operation μ which is coherently associative and unital. Observe that if $\mathcal{O}$ is an operad with multiplication, the spaces $\mathcal{O}(n)$ fit together into a cosimplicial space: if $\nu \in \mathcal{O}(n)$, then

$$d_i\nu = \mu \circ_i \nu \in \mathcal{O}(n+1)$$

and

$$s_i\nu = 1 \circ_i \nu \in \mathcal{O}(n-1),$$

where $\circ_i$ just means "insert in the ith place".

[1] I know I said we can only evaluate formal moduli problems on finite local Artin algebras, but profinite local Artin algebras are totally allowed too.

Example 3.2. Let A be an associative ring. Then the endomorphism operad $\mathrm{End}(A)$ is an operad with multiplication given by the multiplication on A. Moreover, the totalization of the associated cosimplicial abelian group is just the Hochschild complex:

$$\mathrm{Tot}\ (\mathrm{End}(A)^\bullet) \simeq C^*(A, A).$$

The strategy here is to define an operad in posets whose nerve acts on the totalization of any operad with multiplication, then to show that this operad models E_2.

Definition 3.3. The set $W(n)$ of *words of length* n is the set of expressions built from the symbols $\{1, \cdots, n\}$ by formal cup products and compositions.

For instance, $1(5(3), 2, 4 \cup 6)$ is a word of length 6 - call it f. Words of length n form an operad W as n varies by a kind of substitution best illustrated by example: suppose I want to substitute the word $g = 1(2 \cup 3)$ for the letter 5 in the word above. Then I reindex the letters of g so that they begin with 5, adjoin the arguments of g to the arguments of 5 in f[2], and reindex the remaining letters of f to get

$$f \circ_5 g = 1(5(6 \cup 7, 3), 2, 4 \cup 8).$$

And it's a symmetric operad, of course, because we can permute the letters of any word.

There's a nicely Hochschild-looking partial order on each $W(n)$ which is generated by the following requirements:

- $1(2, 3, \cdots, n) > 2 \cup 1(3, \cdots, n)$;
- For each i with $2 \leq i \leq n-1$, we have
$$1(2, 3, \cdots, n) > 1(2, \cdots, i \cup i+1, \cdots, n);$$
- $1(2, 3, \cdots, n) > 1(2, \cdots, n-1) \cup n$;
- The partial orderings are compatible with the operad structure.

With this partial ordering, we claim that the nerve of the operad of posets W is equivalent to the E_2-operad. We won't make the slightest attempt to prove this, but we will draw a picture of $W(2)$:

$$\begin{tikzcd}
 & 1(2) \arrow[dl] \arrow[dr] & \\
1 \cup 2 & & 2 \cup 1 \\
 & 2(1) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

[2]There's a possible problem here concerning the ordering of the arguments of 5. McClure and Smith use a fattened-up version of the operad that allows them to use all possible shuffles between the old arguments and the new arguments.

This is a circle, just as expected.

The final order of business is to describe how W acts on the totalization of an operad with multiplication $\mathcal{O}$. Recall that a point of Tot $\mathcal{O}(\bullet)$ is a list

$$(\cdots, a_2 \in \mathcal{O}(2), a_1 \in \mathcal{O}(1), a_0 \in \mathcal{O}(0))$$

along with all kinds of paths and cells, such as a path in $\mathcal{O}(1)$ between a_1 and $\mu(-, a_0)$.

Suppose we have three such points, $(a_i), (b_i)$ and (c_i), and we wish to feed them into an element of $W(3)$, such as $2(1,3)$. The only thing to do is to pick an operation of the right valency from each list; in this example, we get $b_2(a_0, c_0)$. Note futher that the paths and cells recorded by the points of Tot $\mathcal{O}(\bullet)$ give the paths and cells between operations required by the poset structure of W; for instance, we have a path between $(1(2))((a_i),(b_i)) = a_1(b_0)$ and $(1 \cup 2)((a_i),(b_i)) = \mu(a_0, b_0)$.