JUVITOP 11/03/15

Formality of E_n-operad

E_d is equivalent to the Fulton-Macpherson operad $\{FM_n(d)\}_{n\geq 0}$

$$FM_n(d) = \text{the closure of } C_n(\mathbb{R}^d)/_{\text{dilations \& translations}} \text{ into } \prod_{i\neq j} S^{d-1} \times \prod_{i\neq j\neq k} [0,\infty]$$

$$(x_1,\dots,x_n) \longmapsto \left(\frac{x_i - x_j}{|x_i - x_j|}, \frac{|x_i - x_j|}{|x_i - x_k|}\right)$$

$FM_n(d)$ is a mfld w/ corners and its interior is $C_n(\mathbb{R}^d)/_{\text{dil \& trans}}$

Formality of an operad: formality of $\mathcal{O}_n$'s where the zig-zags respect the operadic structure.

Cohomology of configuration spaces

<u>Thm:</u> $H^*(C_n(\mathbb{R}^d);\mathbb{R})$ can be computed

<u>Proof</u> $(d=2)$ Define forms $\omega_{ij} = d\left(\frac{z_i - z_j}{|z_i - z_j|}\right)$ $\quad (\mathbb{R}^2 = \mathbb{C})$

<u>Step 1:</u> I can explicitly construct cycles in $C_n(\mathbb{R}^d)$ that are the dual basis to ω_{ij}

<u>Step 2:</u> There are the Arnol'd relations

$$\omega_{ij}\omega_{jk} + \omega_{ik}\omega_{jk} + \omega_{ij}\omega_{ik} = 0$$

<u>Ex:</u> This relation is true on the nose

<u>Step 3:</u> Serre spectral sequence shows that this is the only relation

<u>Case $d\geq 3$</u> $\quad \bigvee S^d \to C_{n+1}(\mathbb{R}^d) \to C_n(\mathbb{R}^d)$ shows that the rank of H^{d-1} is $\binom{n}{d}$ again, so we want to

$$\theta_{ij} : C_n(\mathbb{R}^d) \longrightarrow S^{d-1}$$
$$\bar{x} \longmapsto \frac{x_i - x_j}{|x_i - x_j|}$$

$$\omega_{ij} = \theta_{ij}^*(vol_{S^{n-1}})$$

and you can construct cycles detecting these cohomology classes

Step 3: The Arnol'd relation is true only up to cohomology:

$$\sum \omega_{ij}\,\omega_{jk} = d\beta$$

Step 4: Serre SS concludes. □

Q: Is $C_n(\mathbb{R}^d)$ formal?

A: For $d=2$ yes! What about $d>2$? Yes again, but not obvious

We have maps $C_{n+m}(\mathbb{R}^d) \longrightarrow C_n(\mathbb{R}^d)$

In fact θ_{ij} is the case $n=2$ $\quad (C_2(\mathbb{R}^d) \simeq S^{d-1})$

Try pushing forward but we need compact fibers:

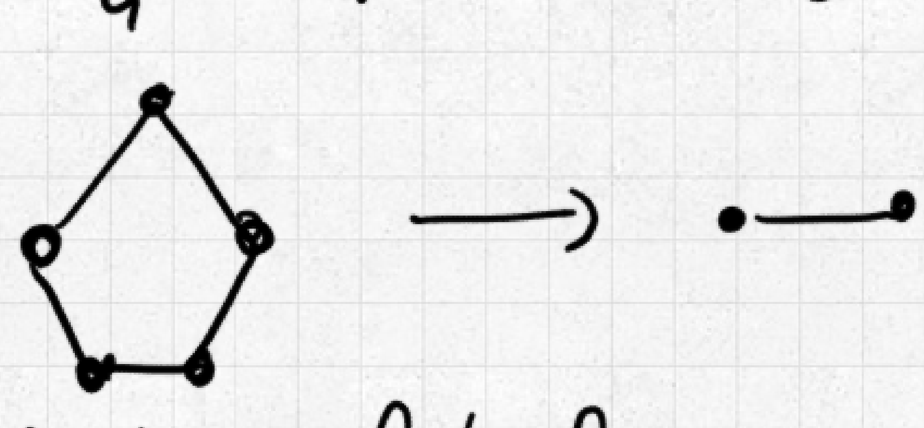
switch to $FM_n(d)$ (which is in fact compact & also an operad)

Def: If a form is in the algebra gen by ω_{ij} we call it an Arnol'd form

Anything that I get by pushing forward is called a generalised Arnol'd form.

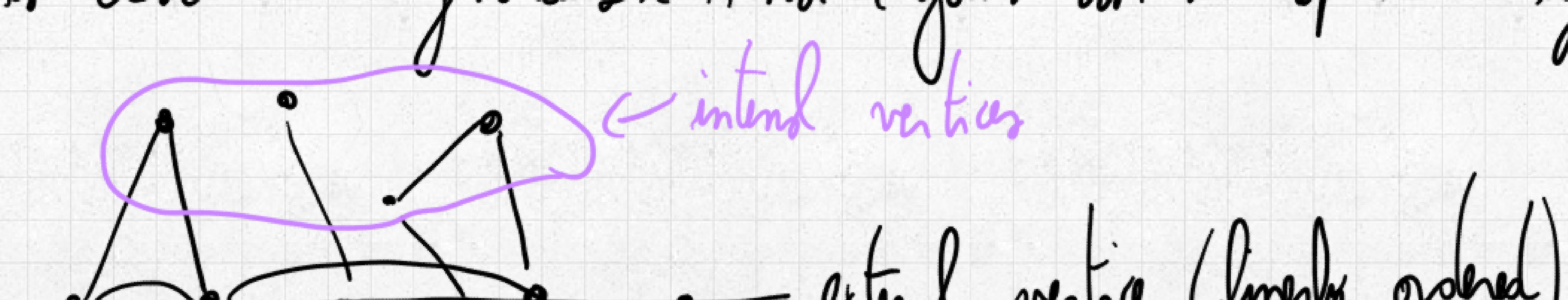
Remark: $FM_{n+m}(\mathbb{R}^d) \longrightarrow FM_n(\mathbb{R}^d)$ are not submersions, but we can make it work (w/ distribution-valued forms or semi-algebraic forms)

Ex: $FM_4(\mathbb{R}) \longrightarrow FM_3(\mathbb{R})$



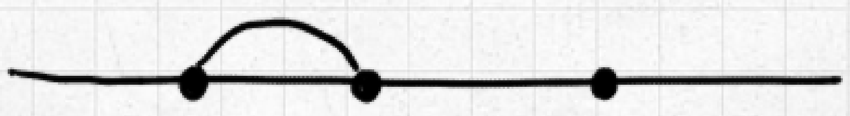
FACT: Generalised Arnol'd forms can be integrated over enough cycles.

It is "clear" that generalised Arnol'd forms can be represented by graphs

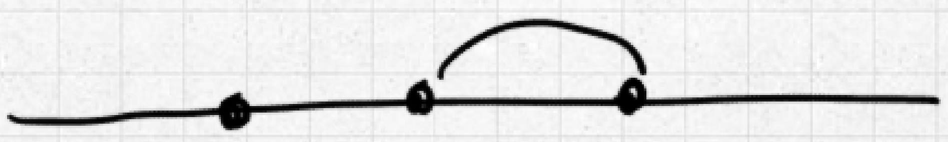


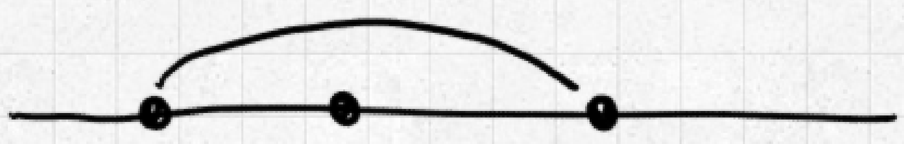
Main observation: GAF are a dg subalgebra (d & $\wedge$ have "graphical" meanings)

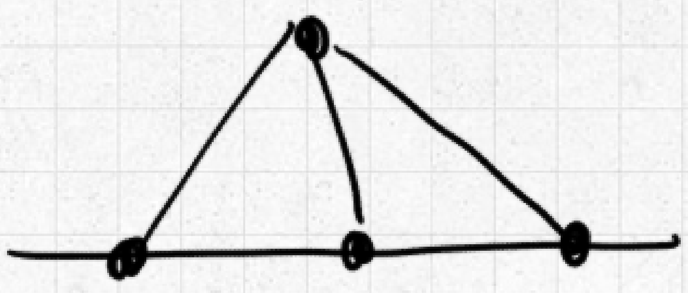
Ex: $FM_3(\mathbb{R}^3)$

$\omega_{12} =$ 

$d\beta = \omega_{12}\omega_{23} + \ldots$

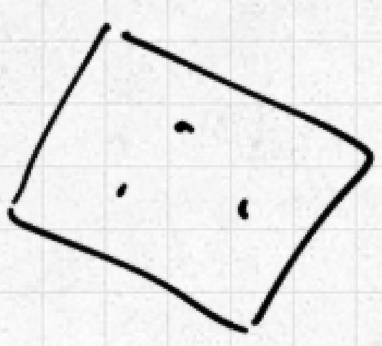
$\omega_{23} =$ 

$\omega_{13} =$ 

$\beta =$ 

$$FM_4(\mathbb{R}^3) \xrightarrow{\pi} FM_3(\mathbb{R}^3)$$

$$\pi\big|_{\partial FM_4(\mathbb{R}^3)} =: \pi_\partial$$

The fiber over  is 

1234 sphere where the 4th pt runs off at ∞

14 24 34 sphere where the 4th point crashes through one of the 3 pts

$\beta := \pi_* \tilde{\beta}$

$\tilde{\beta} = \omega_{14} \wedge \omega_{24} \wedge \omega_{34}$

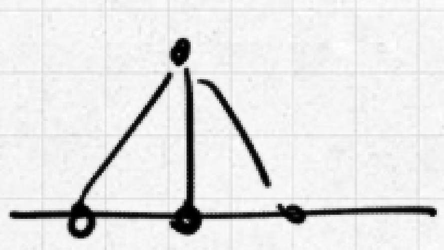
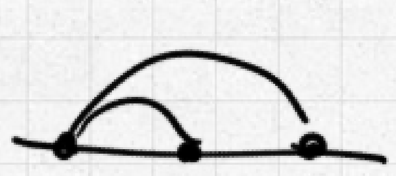
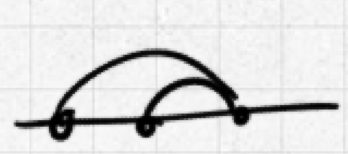
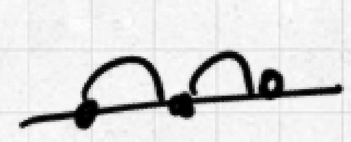
$d\beta = \pi_* d\tilde{\beta} = (\pi_\partial)_* \beta|_\partial$ (Stokes thm!)

$$= \int_{14} \tilde{\beta} + \int_{24} \tilde{\beta} + \int_{34} \tilde{\beta} + \int_{1234} \tilde{\beta}$$

$\int_{14} \tilde{\beta} = \omega_{12} \wedge \omega_{13}$ since ω_{14} is the volume form of the sphere 14

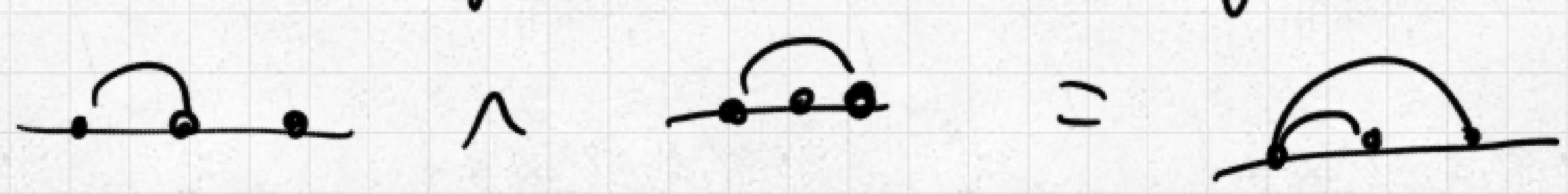
$\omega_{24} \sim \omega_{12}$, $\omega_{34} \sim \omega_{13}$

$\int_{1234} \tilde{\beta} = 0$ since $\omega_{14} \sim \omega_{24} \sim \omega_{34}$ on 1234

$\Rightarrow d$  $=$  $+$  $+$ 

$d(\text{graph}) =$ sum of contractions over all edges

$\wedge$ is stick two graphs together along

Ex: 

Rmk: Some of these forms are unexpectedly 0
(when an internal vertex has valency ≤ 2, if there's a double edge, ...)

Form an ideal out of these vanishing forms & mod out $\Rightarrow$ algebra of admissible diagrams. Call it Graphs_n $\leftarrow$ # of external vertices

Graphs_n also form a cooperad in cdga

Stable formality

$X \longmapsto C_*(X)$ is monoidal

$\Rightarrow \{C_*(\mathcal{O}(n))\}_{n\geq 0}$ is an operad for any operads

$\mathcal{O}(n)$ is stably formal $\iff$ $C_*(\mathcal{O}(n))$ it's q-iso to $H_*(\mathcal{O}(n))$

Claim: FM is stably formal

$$\begin{tikzcd} SA_*(FM_d(n)) \arrow[r, "\sim"] \arrow[d, "\wr"] & C_*(FM_d(n)) \\ (\mathrm{Graphs}_n)^\vee \\ \mathrm{Forests}_n^\vee \arrow[u, hook, "\wr"'] \end{tikzcd}$$

(semi-algebraic chains)

$\leftarrow$ use the Arnol'd relation

$\leftarrow$ no cycles

Formality:

- We would really like to take the deRham complex, but this doesn't work because my forms are not smooth

Also $X \longmapsto \Omega_{dR}(X)$ is colax-monoidal

$$\begin{tikzcd} & \mathsf{Graphs}_n \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ H^*(\Omega_{SA}(FM_n(d))) & & \Omega_{SA}(FM_d(d)) \end{tikzcd}$$

map implementing the computation of cohomology of $C_n(\mathbb{R}^d)$.