JUVITOP 03/03/15

Fulton-MacPherson operad & configuration spaces

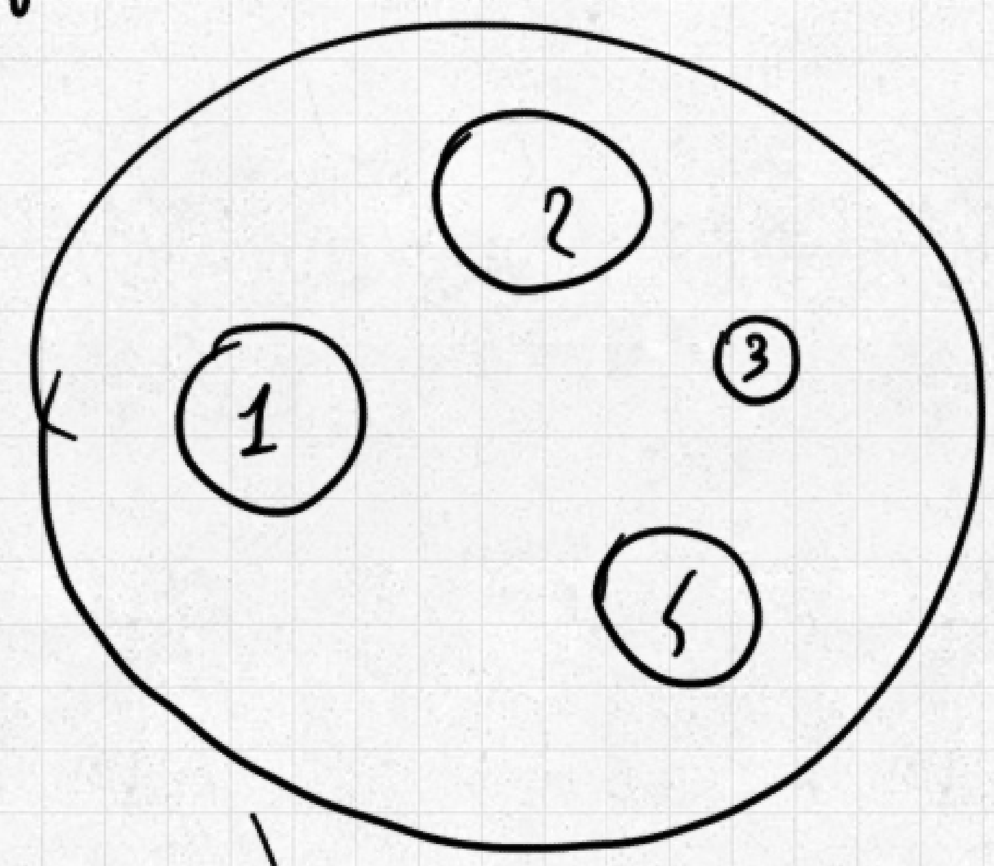
<u>Def</u>: An <u>operad</u> is a sequence of spaces $\mathcal{C}(n)$ that behave like $\mathrm{Map}(X^n, X)$

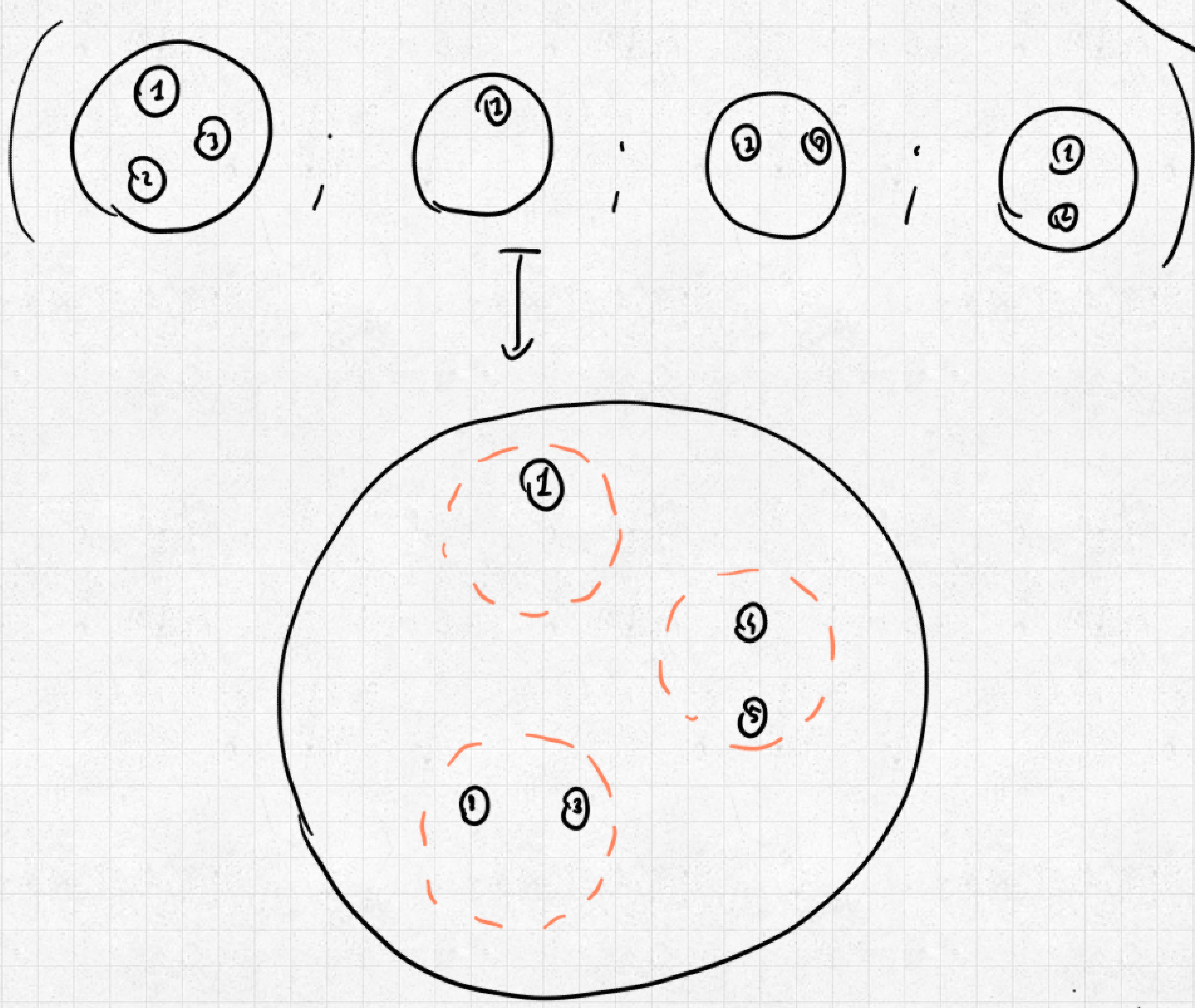
- $\mathcal{C}(n)\times\mathcal{C}(e_1)\times\cdots\times\mathcal{C}(e_n)\longrightarrow\mathcal{C}(e_1+\cdots+e_n)$
- $\mathbb{1}\in\mathcal{C}(1)$
- Σ_n action on $\mathcal{C}(n)$

and these things have to satisfy some conditions

Little N-discs operad

$$\mathcal{C}_N(j)=\mathrm{Emb}\left((D^N)^{\amalg j}, D^N\right)$$





<u>Def</u>: $\mathrm{Conf}_k(\mathbb{R}^N)=\{(x_1,\dots,x_k)\in(\mathbb{R}^N)^k \mid x_i\neq x_j \text{ if } i\neq j\}$

<u>Obs</u>: $\mathcal{C}_N(k)$ is weakly equivalent to $\mathrm{Conf}_k(\mathbb{R}^N)$

⚠ $\mathrm{Conf}_k(\mathbb{R}^N)$ do not form an operad.

The Fulton-MacPherson operad fixes that.

Little modification: we want to mod out by translation and dilations

<u>Ex</u>: $\mathrm{Conf}_2(\mathbb{R}^N)=\{(x,y)\in(\mathbb{R}^N)^2 \mid x\neq y\}\simeq\{(x,y)\in(\mathbb{R}^N)^2 \mid x+y=0,\ |x|=|y|=1\}\cong S^{N-1}$

Things get weird when you add another point

$\mathrm{Conf}_3(\mathbb{R}^2) = ?$ We need to introduce a notion of relative distance

$$\frac{|x_2-x_3|}{|x_1-x_2|} =: \text{"relative distance"}$$

<u>Redundant way to specify configurations</u>

$(x_1,\dots,x_5)\in(\mathbb{R}^N)^5$

directions relative distance

$$\mathrm{Conf}_k(\mathbb{R}^N) \xhookrightarrow{\iota_k} \prod_{i\neq j} S^{N-1}\times \prod_{i,j,k \text{ distinct}} [0,\infty]$$

$$(x_1,\dots,x_k)\longmapsto \left(\frac{x_i-x_j}{|x_i-x_j|}, \frac{|x_i-x_j|}{|x_k-x_j|}\right)$$

<u>Def</u>: The Fulton-Macpherson operad $F_N(k) = \overline{\iota_k(\mathrm{Conf}_k(\mathbb{R}^N))}$

<u>Remark</u>: So we may have coinciding points but we remember the "direction" spanned by them.

This in fact has an operad structure.

<u>Thm</u>: $\mathcal{C}_N(k)\simeq F_N(k)$

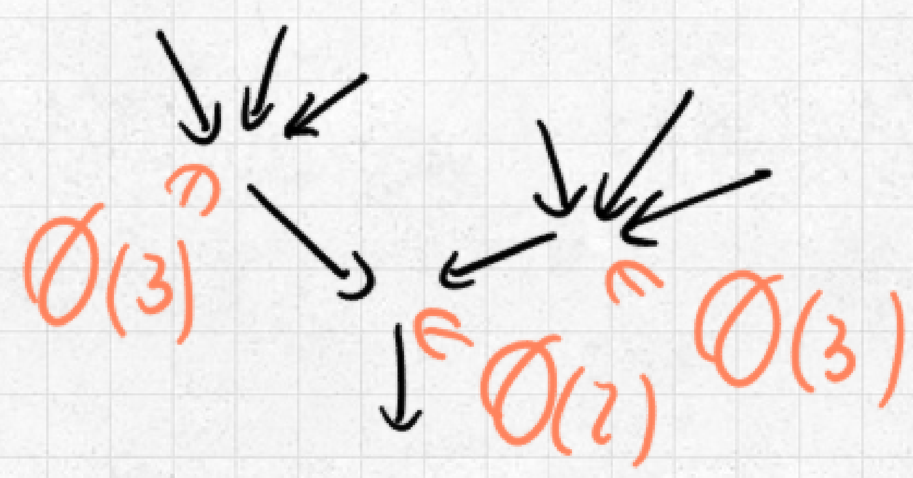
<u>Proof</u>: $\mathcal{C}_N(k) \xleftarrow{\sim} W\mathcal{C}_N(k) \xrightarrow{\sim} F_N(k)$

Boardman-Vogt W construction

<u>Def</u>: Given $\mathcal{O}$ an operad you can define $W\mathcal{O}$ as

$W\mathcal{O}(k)$ = trees s.t. every vertex has a label by an element of $\mathcal{O}$ of the same arity

inner edges have distances $\in[0,1]$
and there are k leaves



$$W\mathcal{O}(k) = \left(\coprod_{\text{tree shapes}} \prod_{\text{inner vertices}} \mathcal{O}(-)\times\prod_{\text{edges}}[0,1]\right)\Big/ \text{collapse edges of length 0.}$$

Claim: For every operad $W\mathcal{O} \xrightarrow{\simeq} \mathcal{O}$ is a weak equivalence.

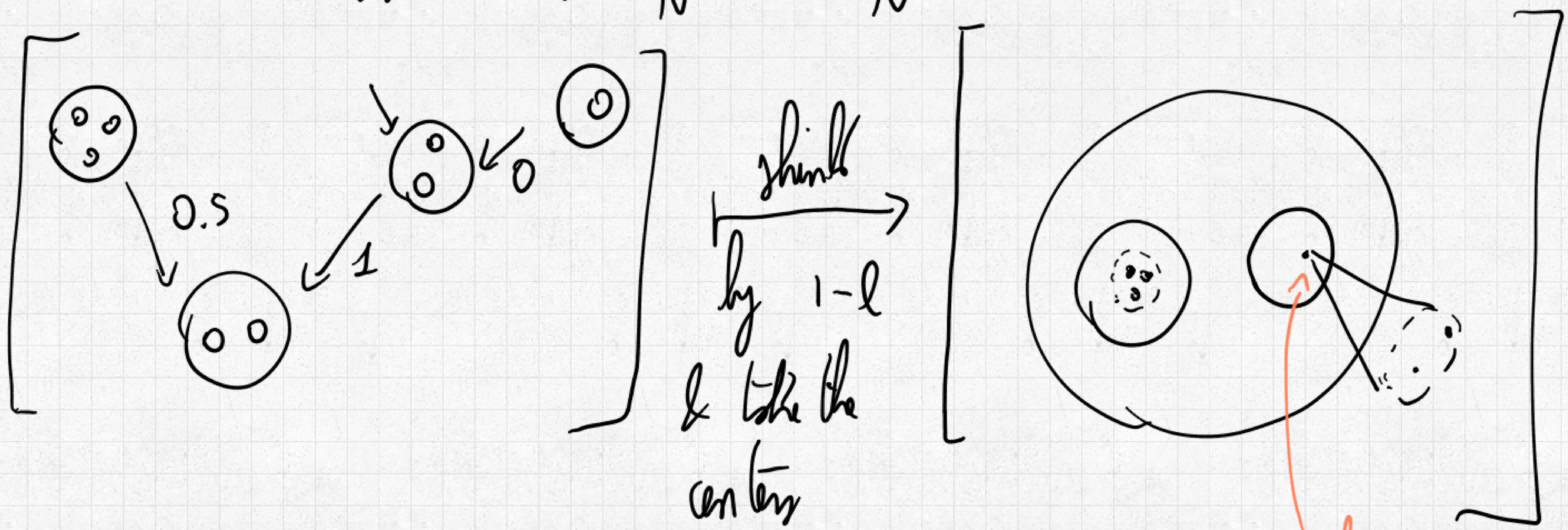
(IDEA: Slowly "shorten all edges"

$$h_t : W\mathcal{O} \longrightarrow \mathcal{O}$$

multiply by t the length of every edge

$\mathcal{O} \subseteq W\mathcal{O}$ (trees of height 1) deformation retract)

Now we want to construct $WC_N \to F_N$



make points ∞-ly close.

(define it for length < 1 & then extend by continuity),

Configuration spaces

There's a fiber bundle

$$\begin{array}{ccc} \mathbb{R}^N \setminus \binom{k-1}{\text{pts}} \longrightarrow \mathrm{Conf}_k(\mathbb{R}^N) & & (x_1, \dots, x_k) \\ \downarrow p & & \downarrow \\ \mathrm{Conf}_{k-1}(\mathbb{R}^N) & & (x_1, \dots, x_{k-1}) \end{array}$$

p has a section $(x_1, \dots, x_{k-1}) \xmapsto{s} (x_1, \dots, x_{k-1}, (|x_1| + \dots + |x_{k-1}| + 1)e_1)$

Corollary: $\mathrm{Conf}_k(\mathbb{R}^N)$ is $(N-2)$-connected

(since $\mathbb{R}^N \setminus \binom{k-1}{\text{pts}} \simeq (S^{N-1})^{\vee(k-1)} \hookleftarrow (N-2)$-connected & long exact sequence in π_*)

Cor: So $\mathrm{Conf}_k(\mathbb{R}^N)$ is $(N-2)$-connected & its homology is
$$H_{N-1}(\mathrm{Conf}_k(\mathbb{R}^N)) \simeq \mathbb{Z}^{\binom{k}{2}}$$

Proof: LES in π_k is split since we have sections + Hurewicz thm. $\square$

Examples of cohomology classes

$$\mathrm{Conf}_k(\mathbb{R}^N) \xrightarrow{\alpha_{i,j}} S^{N-1}$$
$$(x_1, \dots, x_k) \longmapsto \frac{x_i - x_j}{|x_i - x_j|}$$

$$\alpha_{i,j}^* : H^*(S^{N-1}) \longrightarrow H^*(\mathrm{Conf}_k(\mathbb{R}^N))$$
$$[S^{N-1}] \longmapsto a_{i,j}$$

Examples of homology classes

Fix $\varepsilon > 0$ $\quad P_{i,j} \subseteq \mathrm{Conf}_k(\mathbb{R}^N)$ submanifold obtained by fixing
$x_n \in D^N$ and $P_{i,j} = \{ x_i \in S^{N-1}, x_j = -x_i \} \cong S^{N-1}$

$$\Rightarrow \quad a_{i,j}(P_{k,l}) = \begin{cases} 1 & \text{if } (i,j) = (k,l) \\ 0 & \text{otherwise} \end{cases}$$

$a_{i,j}(P_{k,l})$ is a certain map $S^{N-1} \longrightarrow S^{N-1}$ which can be explicitly described and is obviously contractible if $(i,j) \neq (k,l)$.

$\Rightarrow P_{k,l}$ is a basis for $H_{N-1}(\mathrm{Conf}_k(\mathbb{R}^N))$.

- $a_{i,j} = (-1)^N a_{j,i}$ (obvious)
- Arnold relation
$$a_{ij} a_{jk} + a_{jk} a_{ki} + a_{ki} a_{ij} = 0$$

Sketch proof:

1) Reduce to $Conf_3$

2) Look at the submanifold $\{(x_1, x_2, x_3) \text{ collinear}\}$

3) This has 3 connected components

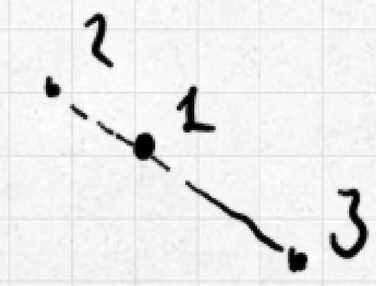
C_i = submanifold where i is in the middle.

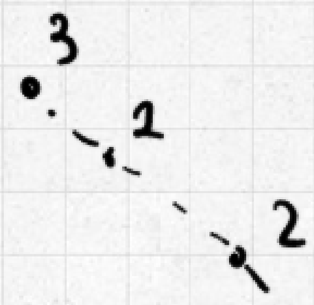
4) C_i^* = cohomology class represented by C_i

$C_1^* C_2^* = 0$

5) what is C_i^* ?

$C_1 \cap P_{2,3} = \emptyset$

$C_1 \cap P_{1,3} = \{$  $\}$

$C_1 \cap P_{1,2} = \{$  $\}$

opposite signs

So $C_1^* = \pm(a_{12} - a_{13})$

and from (4) we're done.

• $a_{ij} a_{kl} = (-1)^? a_{kl} a_{ij}$

<u>Thm</u>: $H^*(Conf_k(\mathbb{R}^N))$ is generated by the Arnold relations.